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The Probability of Intersection of Independent Random Walks in Four Dimensions*

Gregory F. Lawler

Department of Mathematics, Duke University, Durham, NC 27706, USA

Abstract. Let S_1 and S_2 be independent simple random walks of length n in Z^4 starting at 0 and x_0 respectively. If $|x_0|^2 \approx n$, it is shown that the probability that the paths intersect is of order $(\log n)^{-1}$. If $x_0 = 0$, it is shown that the probability of no intersection of the paths decays no faster than $(\log n)^{-1}$ and no slower than $(\log n)^{-1/2}$. It is conjectured that $(\log n)^{-1/2}$ is the actual decay rate.

1. Introduction

Let $S_1(n, \omega)$ and $S_2(n, \omega)$ be independent simple random walks in Z^4 starting at 0 and x_0 respectively; that is, S_1 and S_2 are independent processes indexed by the nonnegative integers satisfying:

(i) $S_1(0, \omega) = 0$ a.s. (almost surely),
(ii) $S_2(0, \omega) = x_0$ a.s.,
(iii) for each $x \in Z^4$, $e \in Z^4$, $|e| = 1$,

$$P\{S_i(n+1, \omega) - S_i(n, \omega) = e \mid S_i(n, \omega) = x\} = 1/8\,.$$

Let $\Pi_i(m, n, \omega)$ denote the random set

$$\Pi_i(m, n, \omega) = \{S_i(k, \omega) : m < k \leqq n\}\,.$$

In understanding the interaction of random particles, it is useful to understand the behavior of $\Pi_1(0, n) \cap \Pi_2(0, m)$. In [4], it was shown that with probability one,

$$\Pi_1(0, \infty) \cap \Pi_2(0, \infty) \neq \emptyset$$

(this is not true for simple random walk in Z^d, $d \geqq 5$). It is well known [1], however that if $W_1(s)$ and $W_2(t)$ are independent Wiener processes taking values in R^d and $\Gamma_i(0, s) = \{W_i(r) : 0 < r \leqq s\}$, then almost surely,

$$\Gamma_1(0, \infty) \cap \Gamma_2(0, \infty) = \emptyset\,,$$

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