Commun. Math. Phys. 83, 261–280 (1982)

Communications in
Mathematical
Physics

Time-Orthogonal Unitary Dilations and Noncommutative Feynman-Kac Formulae

R. L. Hudson*[1], P. D. F. Ion[2], and K. R. Parthasarathy*[3]

1 Mathematics Department, University of Nottingham, University Park, Nottingham NG7 2RD, England

2 Mathematical Reviews, University of Michigan, 611 Church Street, Ann Arbor, MI 48109, USA

3 Indian Statistical Institute, 7, S.J.S. Sansanwal Marg, New Dehli-110016, India

Abstract. An analysis of Feynman-Kac formulae reveals that, typically, the unperturbed semigroup is expressed as the expectation of a random unitary evolution and the perturbed semigroup is the expectation of a perturbation of this evolution in which the latter perturbation is effected by a cocycle with certain covariance properties with respect to the group of translations and reflections of the line. We consider generalisations of the classical commutative formalism in which the probabilistic properties are described in terms of non-commutative probability theory based on von Neumann algebras. Examples of this type are generated, by means of second quantisation, from a unitary dilation of a given self-adjoint contraction semigroup, called the time orthogonal unitary dilation, whose key feature is that the dilation operators corresponding to disjoint time intervals act nontrivially only in mutually orthogonal supplementary Hilbert spaces.

1. Introduction

For any positive-self-adjoint operator H in a Hilbert space $\mathscr{h}_0$ we construct a *time-orthogonal unitary dilation* $\{U_{s,t}, s \geqq t\}$ of the contraction semigroup $\{e^{-tH}, t \geqq 0\}$ in the Hilbert space $\mathscr{h}_0 \oplus L_2(\mathbb{R}, \mathscr{h}_0)$ which exhibits the following properties:

(1) the family $\{U_{s,t}, s \geqq t\}$ is a strongly continuous unitary evolution in the sense that $U_{r,s}U_{s,t} = U_{r,t}$ for all $r \geqq s \geqq t$;

(2) it is time-orthogonal in the sense that $U_{s,t}$ acts nontrivially only in the subspace $\mathscr{h}_0 \oplus \mathscr{h}_{]s,t]}$ where $\mathscr{h}_{]s,t]} \subset L_2(\mathbb{R}, \mathscr{h}_0)$ is the subspace of functions with support contained in $]s,t]$;

(3) it is covariant under time translations and time reversal. This dilation is somewhat different from the classical dilations [14] of a one parameter con-

* Parts of this work were carried out when the first author was a participant in the Sonderforschungsbereich 123 (Stochastische Mathematische Modelle) of the Deutsche-Forschungsgemeinschaft and when the third author was Visiting Fellow at the University of Warwick

0010-3616/82/0083/0261/$04.00