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An Inequality for Hilbert-Schmidt Norm

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Abstract. For the absolute value $|C|=(C^*C)^{1/2}$ and the Hilbert-Schmidt norm $\|C\|_{\mathrm{HS}}=(\operatorname{tr} C^*C)^{1/2}$ of an operator C, the following inequality is proved for any bounded linear operators A and B on a Hilbert space

$$\| |A|-|B| \|_{\mathrm{HS}} \leqq 2^{1/2} \|A-B\|_{\mathrm{HS}}.$$

The corresponding inequality for two normal states φ and ψ of a von Neumann algebra $\mathbf{M}$ is also proved in the following form:

$$d(\varphi, \psi) \leqq \|\xi(\varphi)-\xi(\psi)\| \leqq 2^{1/2} d(\varphi, \psi).$$

Here $\xi(\chi)$ denotes the unique vector representative of a state χ in a natural positive cone $\mathscr{P}^\natural$ for $\mathbf{M}$, and $d(\varphi, \psi)$ denotes the Bures distance defined as the infimum (which is also the minimum) of the distance of vector representatives of φ and ψ. In particular,

$$\|\xi(\varphi_1)-\xi(\varphi_2)\| \leqq 2^{1/2} \|\xi_1-\xi_2\|$$

for any vector representatives ξ_j of φ_j, $j=1,2$.

1. Main Results

In a study of quasi-equivalence of quasifree states of canonical commutation relations, we have encountered the following inequality, which seems to have an independent interest and hence we present it here as an independent article.

Theorem 1. *For any two bounded linear operators A and B on a Hilbert space* $\mathbf{H}$,

$$\| |A|-|B| \|_{\mathrm{HS}} \leqq 2^{1/2} \|A-B\|_{\mathrm{HS}}. \tag{1.1}$$

Remark. The coefficient $2^{1/2}$ is the best possible for a general A and B. If A and B are restricted to be selfadjoint, then the best coefficient is 1 instead of $2^{1/2}$. (Lemma 5.2, [1].)

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