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# On the Existence of Feigenbaum's Fixed Point

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**Abstract.** We give a proof of the existence of a $\mathscr{C}^2$, even solution of Feigenbaum's functional equation

$$g(x) = -\lambda_0^{-1} g(g(-\lambda_0 x)),\ g(0) = 1,$$

where $g$ is a map of $[-1, 1]$ into itself. It extends to a real analytic function over $\mathbb{R}$.

## 1. Introduction

In this paper we give the details of the work described in [1] by the same authors and Ruelle. While the latter is not responsible for possible mistakes in the present paper, he is, of course, a coauthor of the rest. Our purpose is to prove the existence of a $\mathscr{C}^2$ solution of Feigenbaum's functional equation.

$$\begin{cases} g(x) = -\dfrac{1}{\lambda_0} g(g(-\lambda_0 x)), \\ g(0) = 1, \end{cases} \tag{1}$$

where $g$ is a map of the interval $[-1, 1]$ into itself. This equation and its solution play an important role in the theory, initiated by Feigenbaum [7] concerning universal properties of one-parameter families of maps of the interval. Excellent introductions to this theory can be found in [7–9] and particularly in several works of Collet, Eckmann, Koch, and Lanford [2–5], so that we shall give no further details. It is important to note that, to each $\varepsilon > 0$, corresponds the problem of finding a solution of (1) behaving, for small $|x|$, like $1 - \text{const}|x|^{1+\varepsilon}$: for sufficiently small $\varepsilon$ the problem has been fully and satisfactorily solved by Collet et al. [5], who show that there is an $\varepsilon$-dependent solution $g_\varepsilon = f_\varepsilon(|x|^{1+\varepsilon})$, $f_\varepsilon$ analytic.

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