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The Boltzmann Equation with a Soft Potential

I. Linear, Spatially-Homogeneous

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Abstract. The initial value problem for the linearized spatially-homogeneous Boltzmann equation has the form $\frac{\partial f}{\partial t}+Lf=0$ with $f(\boldsymbol{\xi}, t=0)$ given. The linear operator L operates only on the ξ variable and is non-negative, but, for the soft potentials considered here, its continuous spectrum extends to the origin. Thus one cannot expect exponential decay for f, but in this paper it is shown that f decays like $e^{-\lambda t^{\beta}}$ with $\beta<1$. This result will be used in Part II to show existence of solutions of the initial value problem for the full nonlinear, spatially dependent problem for initial data that is close to equilibrium.

1. Introduction

The initial value problem for the Boltzmann equation of kinetic theory is

$$\frac{\partial F}{\partial t}+\boldsymbol{\xi}\cdot\frac{\partial F}{\partial \mathbf{x}}+Q(F,F)=0, \qquad F(t=0)=F_0 \tag{1.1}$$

in which

$$F=F(\boldsymbol{\xi}, t, \mathbf{x}), \tag{1.2}$$

$$t\in\mathbb{R}^+, \quad \boldsymbol{\xi}\in\mathbb{R}^3, \quad \mathbf{x}\in\mathbb{R}^3. \tag{1.3}$$

Throughout this paper a boldface letter will represent a vector in $\mathbb{R}^3$, while the non-boldface letter signifies its magnitude. The quadratically nonlinear operator Q vanishes if F is a Maxwellian:

$$F_M=\frac{\varrho}{(2\pi T)^{3/2}}e^{-|\boldsymbol{\xi}-\mathbf{u}|^2/2T}, \tag{1.4}$$

where $\varrho, \mathbf{u}, T$ can be any functions of x and t. If they are constants, F_M is an equilibrium solution of (1.1). We will study solutions of (1.1) which are close to such an equilibrium and which are *independent of space*.

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