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On the $1/n$ Expansion

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Abstract. The $1/n$ expansion is considered for the n-component non-linear σ-model (classical Heisenberg model) on a lattice of arbitrary dimensions. We show that the expansion for correlation functions and free energy is asymptotic, for all temperatures above the spherical model critical temperature. Furthermore, the existence of a mass gap is established for these temperatures and n sufficiently large.

1. Introduction

It was noted by Stanley in 1967 [1] that certain lattice spin systems exhibit considerable simplification as n, the number of spin components, becomes large. In fact formally, as $n \to \infty$, these models become the so-called spherical model, introduced and solved by Berlin and Kac in 1952 [2]. Their work should in fact be considered the origin of the studies of large n behaviour of multicomponent systems; it also provided motivation for Stanley's work.

In 1973 Wilson [3] in the context of quantum field theory and Abe [4] and Brezin-Wallace [5], in the context of spin systems, found that there is a systematic way to expand in powers of $1/n$. Subsequently these "$1/n$ expansions" were used to compute a variety of objects of interest, such as critical temperatures and exponents.

Soon several other theories came into the realm of $1/n$ expansion. 1974 't Hooft [6] gave the solution of 2-dimensional QCD with SU(N) gauge group, as $N \to \infty$, and Gross and Neveu [7] studied the $1/N$ expansion for 2-dimensional four-fermion interactions (these had already been touched in Wilson's work). More recently the $1/n$ expansion has been applied ([8, 9]) to $\mathbb{C}P^n$ and related σ-models and it has been also suggested to be useful in 4-dimensional QCD [10].

There are several reasons why the $1/n$ expansion has aroused such an interest. It is a non-perturbative expansion, typically each term being a (formal) sum of infinitely many orders of ordinary perturbation theory. Already the zeroth order

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