Commun. Math. Phys. 72, 119–130 (1980)

Communications in
Mathematical Physics

Constellations and Projective Classical Groups

H. Bacry

Centre de Physique Théorique, Section 2, CNRS Marseille
Faculté des Sciences, Luminy, F-13288 Marseille, France

Abstract. The constellation concept is recalled (geometrical description of a ray in a vector space). The groups $\mathrm{PO}(n+1, \mathbb{C})$ or $P\,\mathrm{Sp}(n+1, \mathbb{C})$ are shown to preserve "harmonic conjugation" between two constellations. The action of the Lorentz subgroup and its rotation subgroup is described. Finally, a theorem concerning Clebsch-Gordan product of constellations is proved.

Introduction

The concept of *constellation* has been introduced a few years ago [1] as a convenient geometrical tool to classify orbits[1] of the rotation group SO(3) acting on states of spin $\frac{n}{2}$, i.e. on rays of the $(n+1)$-dimensional Hilbert space or the projective space $P_n(\mathbb{C})$. Each state of spin $\frac{n}{2}$ can be represented by a constellation of order n on the sphere S^2, that is by a set of n points – not necessarily distinct – on S^2 (this generalizes the well known property valid for $n=1$).

Constellations[2] on S^2 have many applications [1–6], the sphere S^2 having various interpretations, namely $P_1(\mathbb{C})$ or Riemann sphere, the Poincaré sphere [7, 2] (set of polarization states of an electromagnetic plane wave), the set of polarization states of the electron, the Bloch sphere [8, 2, 6] or the celestial sphere itself for which the word *constellation* is self justified.

According to the Klein Erlangen programm [9], *the geometry of constellations must involve some group*. Obviously for the spin states the group is the rotation group SO(3). [The action of SO(3) on S^2 is the trivial action.] For the

1 A mistake has been found in this classification by Michel (private communication). The results of reference 1 must be modified as follows: the orbit $\mathrm{SO}(3)/T$ is present in all representations of integral spin except spins 0, 1, and 3. The mistake was due to the fact that I forgot to take into account the possibility of interlacing octahedrons and tetrahedrons having T as a symmetry group

2 The word *constellation* has been suggested by A. Grossmann and appeared for the first time in [2]

0010-3616/80/0072/0119/$02.40