Commun. Math. Phys. 68, 275–289 (1979)

Communications in
Mathematical
Physics

Convergent and Asymptotic Iteration Methods in General Relativity

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Abstract. We show that the fast motion iteration method in General Relativity gives an asymptotic approximation to exact solutions of the reduced Einstein equations. Rigorous estimates of the error commited at each step of the iteration are derived.

1. Introduction

In General Relativity powerful theorems are known, which guarantee existence and uniqueness of solutions to Einstein's equations [1–4] under very general conditions. In contrast to this, however, very little is known about the validity of the approximation methods, on which the comparison between theory and observations is based. One has in fact reasons to be rather sceptical about the usual procedures.

The main result of this paper is an exact estimate of the error one makes by solving the reduced Einstein equations in the usual fast motion iteration scheme.

We outline the basic arguments with the following model problem: Solve, for a given source $\varrho(x,y,z,t)$, and for a finite time interval $0 \leqq t \leqq T$, the *quasilinear* equation

$$\Box_\phi \phi := (\Box + \phi \partial_x^2)\phi = \varrho\,, \qquad \Box = -\partial_t^2 + \partial_x^2 + \partial_y^2 + \partial_z^2 \tag{1.1}$$

with zero initial data, i.e., $\phi = \partial_t \phi = 0$ for $t = 0$, in which case all properties of the solution should be determined by the source. In the scetch, which we shall give in this introduction, $\| \ \|$ will stand for norms in Banach spaces, which will be specified later in the text. Further "c" will stand for a constant depending only on T.

The existence theorems for the problem (1.1) are based on the iteration

$$\Box_{\phi_n} \phi_{n+1} = \varrho\,, \qquad \phi_0 = 0\,. \tag{1.2}$$

If ϱ is small enough this sequence has the property

$$\|\phi_{n+1} - \phi_n\| \leqq \lambda \|\phi_n - \phi_{n-1}\|\,, \qquad 0 < \lambda < 1\,, \tag{1.3}$$

0010-3616/79/0068/0275/$03.00