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Symmetry and Related Properties via the Maximum Principle

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Abstract. We prove symmetry, and some related properties, of positive solutions of second order elliptic equations. Our methods employ various forms of the maximum principle, and a device of moving parallel planes to a critical position, and then showing that the solution is symmetric about the limiting plane. We treat solutions in bounded domains and in the entire space.

1. Introduction

1.1 In an elegant paper [8], Serrin considered solutions of second order elliptic equations satisfying over-determined boundary conditions. For equations with spherical symmetry he proved that the domain on which the solution is defined is necessarily a ball and that the solution is spherically symmetric. The proof is based on the maximum principle and on a device (which goes back to Alexandroff; see Chap. 7 in [3]) of moving parallel planes to a critical position and then showing that the solution is symmetric about the limiting plane.

In this paper we will use the same technique to prove symmetry of positive solutions of elliptic equations vanishing on the boundary – as well as related results (including some extensions to parabolic equations). Some of the equations we treat are related to physics and our techniques should be applicable to other physical situations. We study solutions in bounded domains and in the entire space. The simplest example in a bounded domain is

Theorem 1. *In the ball $\Omega : |x| < R$ in $\mathbb{R}^n$, let $u > 0$ be a positive solution in $C^2(\bar{\Omega})$ of*

$$\Delta u + f(u) = 0 \quad \text{with} \quad u = 0 \quad \text{on} \quad |x| = R. \tag{1.1}$$

Here f is of class C^1. Then u is radially symmetric and

$$\frac{\partial u}{\partial r} < 0, \quad \text{for} \quad 0 < r < R.$$

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