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On the Rate of Asymptotic Eigenvalue Degeneracy

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Abstract. The gap between asymptotically degenerate eigenvalues of one-dimensional Schrödinger operators is estimated. The procedure is illustrated for two examples, one where the solutions of Schrödinger's equation are explicitly known and one where they are not. For the latter case a comparison theorem for ordinary differential equations is required. An incidental result is that a semiclassical (W–K–B) method gives a much better approximation to the logarithmic derivative of a wave-function than to the wave-function itself; explicit error-bounds for the logarithmic derivative are given.

1. Introduction

The operator

$$\begin{aligned} H(\beta) &= -\frac{d^2}{dx^2} + x^2 - 2\beta x^3 + \beta^2 x^4 \\ &= -\frac{d^2}{dx^2} + x^2(1-\beta x)^2 \end{aligned} \tag{1.1}$$

on $L^2(\mathbb{R})$, with $\beta \geqq 0$, and other similar operators have been studied because they exhibit asymptotic eigenvalue degeneracy as $\beta \to 0$ [1, 2, 3]. The operator $H(\beta)$ is essentially self-adjoint on $\mathscr{S}(\mathbb{R})$. As $\beta \to 0$, it approaches the harmonic oscillator Hamiltonian, $H(0) = -\frac{d^2}{dx^2} + x^2$ (in the strong operator sense), and it is known that two distinct eigenvalues of $H(\beta)$ converge to each eigenvalue of $H(0)$. For example, the two lowest eigenvalues of (1.1), $E_0(\beta)$ and $E_1(\beta)$, both converge to $E_0(0) = 1$ [the Number 1 will frequently be called $E_0(0)$, so that it will be clear how the formulae generalize]; and in fact both $E_0(\beta)$ and $E_1(\beta)$ have the same asymptotic—but almost certainly not convergent—power-series expansions in β,

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