Commun. math. Phys. 59, 79—95 (1978)

Communications in Mathematical Physics

Unbounded Derivations and Invariant States

Ola Bratteli*

UER Scientifique de Marseille-Luminy, and Centre de Physique Theorique CNRS, F-13274 Marseille Cedex 2, France

Uffe Haagerup

Odense Universitet, Matematisk Institut, DK-5230 Odense M, Denmark

Abstract. Let $\mathscr{M}$ be a von Neumann algebra with a cyclic and separating vector Ω. Let $\delta = i[H, \cdot]$ be the spatial derivation implemented by a selfadjoint operator H, such that $H\Omega = 0$. Let Δ be the modular operator associated with the pair $(\mathscr{M}, \Omega)$. We prove the equivalence of the following three conditions:

1) H is essential selfadjoint on $D(\delta)\Omega$, and H commutes strongly with Δ.

2) The restriction of H to $D(\delta)\Omega$ is essential selfadjoint on $D(\Delta^{\frac{1}{2}})$ equipped with the inner product

$$(\xi|\eta)_{\#} = (\xi|\eta) + (\Delta^{\frac{1}{2}}\xi|\Delta^{\frac{1}{2}}\eta), \qquad \xi, \eta \in D(\Delta^{\frac{1}{2}}).$$

3) $\exp(itH)\mathscr{M}\exp(-itH) = \mathscr{M}$ for any $t \in \mathbb{R}$.

We show by an example, that the first part of 1), H is essential selfadjoint on $D(\delta)\Omega$, does not imply 3). This disproves a conjecture due to Bratteli and Robinson [3].

Introduction

In the study of time development of quantum and classical systems one often encounters the following situation ([5, 10, 22] and references given there). The infinite volume Gibbs state ω is specified as a state on C^*-algebra $\mathfrak{A}$ of observables, along with a derivation δ of $\mathfrak{A}$, which should be the derivative at time zero for the time development of the infinite system. This derivation satisfies

$$\omega \circ \delta(x) = 0, \qquad x \in D(\delta).$$

If $(\pi, \mathscr{H}, \Omega)$ is the cyclic representation associated to ω, then there exists a unique symmetric operator H_0, satisfying the properties [3]:

(i) $D(H_0) = \pi(D(\delta))\Omega$,

(ii) $H_0\Omega = 0$,

(iii) $\pi(\delta(x))\xi = i[H_0, \pi(x)]\xi$, $\xi \in D(H_0)$, $x \in D(\delta)$,

---

* Part of this work was done while O.B. was a member of Zentrum für interdisziplinäre Forschung der Universität Bielefeld