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Energy Spectrum of Extremal Invariant States

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Abstract. For an extremal invariant state ω of a weakly asymptotically abelian dynamical system we prove that the corresponding energy spectrum is either one-sided or the whole reals, or a periodic subgroup. The latter case implies abelianness of the algebra in the representation generated by ω.

The purpose of this note is to extend to weakly clustering (i.e. extremal invariant) states of asymptotically abelian systems a spectral alternative useful e.g. in deriving the *KMS* condition from stability assumptions (Prop. 3 of [1], see also Theorem 6.1 of [2]). This result follows from the long known fact that the energy spectrum of clustering states is additive [3]. The generalization we present here is relevant to the description of e.g. crystal states in motion.

Assumptions. $\{\mathfrak{A}, R, \alpha\}$ is the triple of a C^*-algebra $\mathfrak{A}$ together with a strongly continuous one-parameter group $t\in\mathbb{R}\to\alpha_t$ of *-automorphisms of $\mathfrak{A}$. We consider an α-invariant state ω of $\mathfrak{A}$ such that

(i) *(asymptotic abelianness)* for any two $A, B\in\mathfrak{A}$ the commutator $[\alpha_t(A), B]$ tends to zero in mean under all states of the normal folium of ω:

$$\frac{1}{2T}\int_{-T}^{+T}[C[\alpha_t(A),B]D)dt \underset{T\to\infty}{\to} 0, \qquad A,B,C,D\in\mathfrak{A}. \tag{1}$$

(ii) *(weak clustering)*

$$\frac{1}{2T}\int_{-T}^{+T}\omega(A\alpha_t(B))dt \xrightarrow[T\to\infty]{} \omega(A)\omega(B), \qquad A,B\in\mathfrak{A}. \tag{2}$$

Theorem. *Assume the above situation. Let* (π, U) *be the covariant representation of* $\{\mathfrak{A}, R, \alpha\}$, *on the Hilbert space* $\mathscr{H}$ *with cyclic invariant vector* Ω, *generated by the state* ω *via the GNS construction. We have the following alternatives*