Commun. math. Phys. 41, 1—9 (1975)

Inequalities for Ising Models and Field Theories which Obey the Lee-Yang Theorem*

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Received September 4, 1974; in revised form October 9, 1974

Abstract. A series of inequalities for partition, correlation, and Ursell functions are derived as consequences of the Lee-Yang Theorem. In particular, the n-point Schwinger functions of *even* ϕ^4 models are bounded in terms of the 2-point function as strongly as is the case for Gaussian fields; this strengthens recent results of Glimm and Jaffe and shows that renormalizability of the 2-point function by fourth degree counter-terms implies existence of a ϕ^4 field theory with a moment generating function which is entire of exponential order at most two. It is also noted that if *any* (even) truncated Schwinger function vanishes identically, the resulting field theory is a generalized free field.

1. The Lee-Yang Theorem

We consider a collection of (spin) random variables $\{X_j : j=1,\ldots,N\}$ whose moment generating function is of the form,

$$\boldsymbol{E}\left(\exp\left(\sum_j z_j X_j\right)\right) = \frac{\int \exp\left(\sum_j z_j x_j + \sum_{j,k} J_{jk} x_j x_k\right) \prod_j d\varrho_j(x_j)}{\int \exp\left(\Sigma J_{jk} x_j x_k\right) \prod_j d\varrho_j(x_j)}, \tag{1.1}$$

with $J_{jk} \geqq 0$ and each ϱ_j an even probability measure such that $\int \exp(bx^2) d\varrho_j(x) < \infty$ for all real b. (1.1) is the (normalized) partition function for a general Ising model (with two-body ferromagnetic interactions) such as arises in the lattice approximation to even $P(\phi)$ Euclidean field models [1]. Spin-$\frac{1}{2}$ Ising models correspond to letting each $\varrho_j(x) = (\delta(x_j - 1) + \delta(x_j + 1))/2$.

A very general version of the Lee-Yang theorem which applies to spin-$\frac{n}{2}$ as well as to $d\varrho_j/dx = C_j \exp(-a_j x^4 - b_j x^2)$ (corresponding to an even ϕ^4 field theory) is as follows [2, 3, 4]:

Theorem 1. *If for each j, the zeros of $\int \exp(zx) d\varrho_j(x)$ are pure imaginary, then for any choice of $\lambda_j \geqq 0$, the zeros of $\boldsymbol{E}(\exp(z \Sigma \lambda_j X_j))$ are pure imaginary.*

The Lee-Yang theorem has been widely used in both statistical mechanics and quantum field theory [e.g., 5, 6] for the investigation of phase transition phenomena; it will be applied in this paper to obtain correlation inequalities and related results which in the case of field theory should be useful in proving existence and studying triviality (or non-triviality) of ϕ^4 models. We note that perturbation theory predicts the applicability of our results (i.e., Theorem 10 below) in space-time dimension less than 5.

* Supported in part by the Indiana University Foundation and by the National Science Foundation under Grant NSF-GP-24003.