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# Conformal Groups and Conformally Equivalent Isometry Groups

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**Abstract.** It is shown that if an $n$ dimensional Riemannian or pseudo-Riemannian manifold admits a proper conformal scalar, every (local) conformal group is conformally isometric, and that if it admits a proper conformal gradient every (local) conformal group is conformally homothetic. In the Riemannian case there is always a conformal scalar unless the metric is conformally Euclidean. In the case of a Lorentzian 4-manifold it is proved that the only metrics with no conformal scalars (and hence the only ones admitting a (local) conformal group not conformally isometric) are either conformal to the plane wave metric with parallel rays or conformally Minkowskian.

## § 1. Introduction

Yano (1955) has shown that in an $n$ dimensional Riemannian or pseudo-Riemannian manifold $(M_n, g)$ every $r$ dimensional Lie group $C_r$ of local conformal transformations that is *simply transitive* is conformally isometric (§ 3). More recently it has been proved for $n \geqq 4$ (Defrise, 1969; Suguri and Ueno, 1972) that for any manifold $M_n$ with a *positive definite* metric tensor $g_{\mu\nu}$, every (local) conformal group $C_r$ is conformally isometric (except in the case when the metric is conformally flat).

In this paper we present a more general result. We show that if a space $(M_n, g)$ admits a *proper conformal scalar* (in the sense of du Plessis (1969) as described in § 4) every (local) conformal group $C_r$ is conformally isometric; if a space $(M_n, g)$ admits a *proper conformal gradient* then every $C_r$ is conformally homothetic (§ 4).

This suggests the conjecture that the converses of these two theorems are true, i.e. that: a space $(M_n, g)$ with no *proper conformal scalar* admits a conformal group $C_r$ that is not conformally isometric; and that a space with *no proper conformal gradient* admits a proper conformal group (i.e. one that is not conformally homothetic).

These conjectures are easily verified for a positive definite metric by using the theorem (proved by Taub 1951) on the order of the conformal group $C_r$ admitted by conformally flat spaces (§ 4).

These conjectures are also true in the physically interesting case of a Lorentzian manifold. We prove in particular (§ 6) that a Lorentzian 4-manifold with no proper conformal scalar is conformally equivalent to the *plane-wave metric* with *parallel rays* or else conformally *minkowskian.*

The plane-wave metric admits a proper conformal gradient and the conformal group is reduced to a *proper homothety group* $H_6$ or $H_7$.