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Translation Invariant States in Quantum Mechanics

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Abstract. We give a complete description of the states of the C.C.R. algebra for a finite number of degrees of freedom which are invariant with respect to subgroups of the translation group of phase space. We make precise some well-known results of quantum mechanics such as Bloch theorem.

§ I. Introduction

In this paper we shall deal with the algebra $\overline{\Delta(\boldsymbol{G}, \xi)}$ of canonical commutation relations (C.C.R.) introduced in [2] (see also [3]) where $\boldsymbol{G}$ is an abelian group and ξ a bicharacter [Definition (2.1)]; for $\boldsymbol{G} = \mathbb{R}^{2N}$ and $\xi((x,p),(x',p')) = \exp\left(-\frac{i}{2}\sum_{i=1}^{n} x_i p_i' - x_i' p_i\right)$, $(x,p) \in \mathbb{R}^{2N}$, $\overline{\Delta(\mathbb{R}^{2N}, \xi)}$ is the uniform closure of the $*$-algebra of the finite linear combinations of Weyl operators $\delta_{(x,p)}$. The interest of this algebra is that it has a large number of states; namely, any normalized linear positive function on the linear combinations of Weyl operators extends to a (unique) state of $\overline{\Delta(\boldsymbol{G}, \xi)}$.

Translations by $\boldsymbol{g}$ in the phase space $\boldsymbol{G}$ are represented by a $*$-automorphism $\tau_{\boldsymbol{g}}$ of $\overline{\Delta(\boldsymbol{G}, \xi)}$ which is inner:

$$\tau_{\boldsymbol{g}} \delta_{\boldsymbol{g}'} = \delta_{\boldsymbol{g}} \delta_{\boldsymbol{g}'} \delta_{\boldsymbol{g}}^{-1} = \xi(\boldsymbol{g}, \boldsymbol{g}')^2 \delta_{\boldsymbol{g}'}, \qquad \boldsymbol{g}, \boldsymbol{g}' \in \boldsymbol{G}.$$

Let $\boldsymbol{H}$ be a subgroup of $\boldsymbol{G}$ and let $\boldsymbol{H}'$ be the set of elements $\boldsymbol{g}$ of $\boldsymbol{G}$ such that $\xi(\boldsymbol{g}, \boldsymbol{h})^2 = 1$ for any $\boldsymbol{h} \in \boldsymbol{H}$. Then the invariant states ω of $\overline{\Delta(\boldsymbol{G}, \xi)}$ with respect to the $\tau_{\boldsymbol{h}} (\boldsymbol{h} \in \boldsymbol{H})$ are those states for which $\omega(\delta_{\boldsymbol{g}}) = 0$ if $\boldsymbol{g} \notin \boldsymbol{H}'$ [Proposition (2.13)]; conversely, let ω be a state of $\overline{\Delta(\boldsymbol{H}', \xi')}$ where ξ' is the restriction of ξ to $\boldsymbol{H}' \times \boldsymbol{H}'$; then there exists an invariant extension $\bar{\omega}$ of ω to $\overline{\Delta(\boldsymbol{G}, \xi)}$ which is given by $\bar{\omega}(\delta_{\boldsymbol{g}}) = 0$ if $\boldsymbol{g} \notin \boldsymbol{H}'$ [Proposition (2.14)]. Moreover if $\overline{\Delta(\boldsymbol{H}', \xi')}$ is abelian and if ω is pure, $\bar{\omega}$ is the unique (hence pure) extension of ω to $\overline{\Delta(\boldsymbol{G}, \xi)}$ if and only if $\overline{\Delta(\boldsymbol{H}', \xi')}$ is maximal abelian [Proposition (2.18)].