Commun. math. Phys. 38, 1—10 (1974)

On the Equivalence of the *KMS* Condition and the Variational Principle for Quantum Lattice Systems

Huzihiro Araki
Research Institute for Mathematical Sciences, Kyoto University, Kyoto, Japan

Received May 16, 1974

Abstract. For quantum spin systems on a lattice of an arbitrary dimension, the *KMS* condition and the variational principle are shown to be equivalent at an arbitrary temperature for translationally invariant states.

§ 1. Main Result

The *KMS* condition and the variational principle are known to be equivalent for classical spin lattice systems [8]. The equivalence has been shown also for quantum spin lattice systems when either the dimension of the lattice is one or the temperature is high [7]. We shall prove the equivalence for any spin lattice system at arbitrary non-zero temperature.

We use the same notation as in [7]. The assumption on the interaction potential $\Phi(I)$ is as follows:

(i) Translational covariance: $\Phi(I+a)=\tau(a)\,\Phi(I)$.

(ii) Finite-body interaction: $\Phi(I)=0$ if $\mathrm{N}(I)\geqq N_0$ for some N_0.

(iii) Relatively short range: $\|\Phi\|=\sum_{I\ni 0}\|\Phi(I)\|/\mathrm{N}(I)<\infty$.

For a state ψ of the C^*-algebra $\mathfrak{A}$ (of quasi-local operators) and a finite subset Λ of the lattice, ψ_Λ denotes the restriction of ψ to $\mathfrak{A}(\Lambda)$ (the local subalgebra) and ϱ_ψ^Λ denotes the density matrix for ψ_Λ:

$$\varrho_\psi^\Lambda\in\mathfrak{A}(\Lambda)\,,\qquad \psi(Q)=\mathrm{tr}(\varrho_\psi^\Lambda Q)\qquad\text{for all}\qquad Q\in\mathfrak{A}(\Lambda)\,. \tag{1.1}$$

The variational principle at the inverse temperature β is satisfied by a translationally invariant state ψ of $\mathfrak{A}$ if

$$\mathrm{s}(\psi)-\beta\psi(A)=P\equiv\lim_{\Lambda\uparrow}\mathrm{N}(\Lambda)^{-1}\log\mathrm{tr}(e^{-\beta\mathrm{U}(\Lambda)}) \tag{1.2}$$

where $s(\psi)$ is the mean entropy of the state ψ:

$$\mathrm{s}(\psi)=-\lim_{\Lambda\uparrow}\mathrm{N}(\Lambda)^{-1}\,\psi(\log\varrho_\psi^\Lambda)\,, \tag{1.3}$$