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GHS and other Inequalities*

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Abstract. We use a transformation due to Percus to give a simple derivation of the Griffiths, Hurst, and Sherman, and some other new inequalities, for Ising ferromagnets with pair interactions. The proof makes use of the Griffiths, Kelly, and Sherman and the Fortuin, Kasteleyn, and Ginibre inequalities.

1. Introduction

We consider an Ising spin system with ferromagnetic pair interactions; $\sigma_i = \pm 1$, $i \in \Lambda$, $i = 1, \ldots, |\Lambda|$,

$$H(\boldsymbol{\sigma}) = -1/2 \sum_{i \neq j} J_{ij} \sigma_i \sigma_j - \sum_i h_i \sigma_i, \qquad J_{ij} \geqq 0, \tag{1.1}$$

where H is the energy of the system and h_i are external magnetic fields.

The Ursell, or cluster function, $u_l(i_1, \ldots, i_l)$ play a central role in statistical mechanics. They are given for spin systems, by the relations [1]

$$u_l(i_1, \ldots, i_l) = \frac{\partial^l}{\partial h_{i_1} \ldots \partial h_{i_l}} \ln Z(\Lambda; \boldsymbol{h}, \boldsymbol{J}), \qquad l = 1, \ldots, |\Lambda| \tag{1.2}$$

where we have written $\boldsymbol{h}$ and $\boldsymbol{J}$ for the collections $\{h_i\}$, $\{J_{ij}\}$ and

$$Z(\Lambda; \boldsymbol{h}, \boldsymbol{J}) = \mathrm{Tr}_{\{\sigma\}} \exp[-H(\boldsymbol{\sigma})]. \tag{1.3}$$

Thus

$$\begin{aligned} u_1(i) &= \langle \sigma_i \rangle, \qquad u_2(i,j) = \langle \sigma_i \sigma_j \rangle - \langle \sigma_i \rangle \langle \sigma_j \rangle, \\ u_3(i,j,k) &= \langle \sigma_i \sigma_j \sigma_k \rangle - \langle \sigma_i \rangle \langle \sigma_j \sigma_k \rangle - \langle \sigma_j \rangle \langle \sigma_i \sigma_k \rangle \\ &\quad - \langle \sigma_k \rangle \langle \sigma_i \sigma_j \rangle + 2 \langle \sigma_i \rangle \langle \sigma_j \rangle \langle \sigma_k \rangle, \ldots \end{aligned} \tag{1.4}$$

where $\langle \ \rangle$ denotes expectations with respect to the measure $\mu(\boldsymbol{\sigma}) = Z^{-1} \exp[-H]$; (we have set the temperature $\beta^{-1} = 1$).

The Griffiths, Kelly, and Sherman [2] (GKS) inequalities for this system apply when the h_i have the same sign for all i; say $h_i \geqq 0$, $\boldsymbol{h} \geqq 0$, (similar results hold by symmetry when $h_i < 0$, all i). They state

$$\langle \sigma_A \sigma_B \rangle \geqq \langle \sigma_A \rangle \langle \sigma_B \rangle, \qquad \boldsymbol{h} \geqq 0 \tag{1.5}$$

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