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Theory of Monomer-Dimer Systems*

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Abstract. We investigate the general monomer-dimer partition function, $P(x)$, which is a polynomial in the monomer activity, x, with coefficients depending on the dimer activities. Our main result is that $P(x)$ has its zeros on the imaginary axis when the dimer activities are nonnegative. Therefore, no monomer-dimer system can have a phase transition as a function of monomer density except, possibly, when the monomer density is minimal (i.e. $x = 0$). Elaborating on this theme we prove the existence and analyticity of correlation functions (away from $x = 0$) in the thermodynamic limit. Among other things we obtain bounds on the compressibility and derive a new variable in which to make an expansion of the free energy that converges down to the minimal monomer density. We also relate the monomer-dimer problem to the Heisenberg and Ising models of a magnet and derive Christoffell-Darboux formulas for the monomer-dimer and Ising model partition functions. This casts the Ising model in a new light and provides an alternative proof of the Lee-Yang circle theorem. We also derive joint complex analyticity domains in the monomer and dimer activities. Our considerations are independent of geometry and hence are valid for any dimensionality.

I. Introduction

A monomer-dimer system is specified by a graph, G (also called a lattice in the physics literature), together with a family of weights (or Boltzmann factors) assigned to the edges of G. The precise definition of a weighted graph is given in Section II, but for the present we shall assume the reader is familiar with the concept. Dimers can be placed on the edges of G so that no vertex has more than one dimer. Uncovered vertices are called monomers and have a fugacity which we call x. One can also define related problems, such as the monomer-trimer problem, and although the history of these various problems are intertwined we shall consider only the monomer-dimer problem in this paper.

We shall answer the question whether, as the monomer concentration is varied, a phase transition can occur for an infinite system. Our answer,

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