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An Elementary Proof of the Plancherel Theorem for the Classical Groups

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Abstract. We show how to reduce the Plancherel theorem to one lemma which is proved by elementary means.

1. Introduction

The theory of Fourier transformations on Lie groups though being rather young and still incomplete has recently attracted the attention of theoretical physicists. They are interested in particular in the basic theorem on the inversion of the Fourier transformation, the so-called Plancherel theorem. The groups, physicists are mainly concerned with, are the homogeneous Lorentz group $SL(2, C)$, which belongs to the classical simple Lie groups, and the group $SU(1, 1)$ which is non-classical. The proof of the Plancherel theorem for classical groups has first been established by Gelfand and Naimark [1]. In its original form it is extremely tedious. Lateron Gelfand himself found a more elegant proof which exploits the connection between the Plancherel theorem and integral transforms of the Riesz type [2]. Both proofs are repeated in [3][1]. Harish-Chandra [4] dropped most of this dead weight and reduced the proof to a few lemmas. For the convenience of the physicists we undertake in this note to prove the Plancherel theorem for classical groups with a minimal number of elementary arguments, which in particular keep all constant factors under control.

Actually we perform the proof explicitly only for the complex unimodular groups $SL(n, C)$. The orthogonal and symplectic groups can be treated quite the same way. Even for the non-classical group $SU(1, 1)$ the Plancherel theorem can be proved in this fashion [5]. In the case of the homogeneous Lorentz group the correctness of our lemma can directly be inspected and the whole proof of the Plancherel theorem becomes pedestrian. Since our proof is simple even in the case of a general group $SL(n, C)$, we refrain from separately dealing with the case $SL(2, C)$ which is of greatest interest for physicists.

[1] In fact Gelfand proves two Plancherel formulae which differ by a constant factor, compare [3], Eq. 26.83 and ibid. App. III, Eq. 2.42. Our proof agrees with his second result [2].