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On the Converse of the Reeh-Schlieder Theorem

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Abstract. It will be shown that the weak additivity-property is not only sufficient but also necessary for the derivation of the Reeh-Schlieder theorem.

I. Introduction and Results

Many results in quantum field theory which have been derived so far are based directly or indirectly on the Reeh-Schlieder theorem [1]. The central role of this theorem makes it worthwhile to give it separate consideration.

The problem can be formulated as follows. Let $\mathscr{A}$ be a v. Neumann-algebra acting in a Hilbert space $\mathscr{H}$ and $\mathscr{B} \subset \mathscr{A}$ a sub v. Neumann-algebra. Do there exists vectors $x \in \mathscr{H}$ such that $\overline{\mathscr{B}x} = \overline{\mathscr{A}x}$? If yes, how can one characterize such vectors? This problem will hardly be soluable in full generality but there exists a class of v. Neumann algebras for which the answer is partly known. This class is of particular interest for physics.

Here we have a v. Neumann algebra $\mathscr{A}$ together with a n-parametric group G of normal automorphisms which are implemented by a strongly continuous unitary representation of G having its spectrum in a proper closed cone.

We consider G as the additive group of $\boldsymbol{R}^n$ and denote by $U(g)$ its representation in $\mathscr{H}$. A vector $x \in \mathscr{H}$ is called analytic for if $U(g)x$ has an analytic extension into a full neighbourhood of the origin in $\boldsymbol{C}^n$.

Let $\mathscr{N}$ be any open neighbourhood of the origin in $\boldsymbol{R}^n$ and $\mathscr{B}$ a sub-algebra of $\mathscr{A}$ then we denote by $(\mathscr{B}, \mathscr{N})$ the v. Neumann algebra generated by $\{U(g)\,\mathscr{B}\,U(g^{-1});\, g \in \mathscr{N}\}$.

With these notations we get:

1. Theorem (Reeh-Schlieder). *With $\mathscr{A}$, G and $U(g)$, $g \in G$ as described above, assume $\mathscr{B} \subset \mathscr{A}$ is a sub v. Neumann algebra of $\mathscr{A}$ such that*

$$(\mathscr{B}, G) = \Big\{\bigcup_{g \in G} U(g)\,\mathscr{B}\,U(g^{-1})\Big\}'' = \mathscr{A}\,.$$

Then for any vector $x \in \mathscr{H}$ which is analytic for $U(g)$, and for any open neighbourhood $\mathscr{N}$ of the origin in G, we have the relation

$$\overline{(\mathscr{B}, \mathscr{N})x} = \overline{\mathscr{A}x}\,.$$