Commun. math. Phys. 7, 222—224 (1968)

Sequential Convergence in the Dual of a W^*-Algebra

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Received November 19, 1967

Abstract. The present paper is the result of the author's attempt to extend Theorem 9 of [5] to the case of a non-abelian W^*-algebra. In [5] GROTHENDIECK proves that weak and weak* convergence are equivalent for sequences in the dual space of an abelian W^*-algebra. Theorem 4 of the present paper is only a partial result in that direction, but it is presented here because of its possible worth as a technical tool.

I. Preliminaries and Notation

Let A be a W^*-algebra. By [8, p. 1.74] the second dual A^{**} of A is also a W^*-algebra, and we shall consider the canonical imbedding of A into A^{**} as an identification. By [2, p. 126] there exists a central projection $z \in A^{**}$ which is the supremum of the set of minimal projections in A^{**}. Set $z' = 1 - z$; let $A_d^* = \{f \in A^* : f \mid z'A^{**} = 0\}$, and $A_c^* = \{f \in A^* : f | zA^{**} = 0\}$. Since z is central, $A^* = A_c^* \oplus A_d^*$, and both A_c^* and A_d^* are closed invariant subspaces [7, p. 439] of A^*. Thus by [7, p. 439] any positive $f \in A^*$ has a unique decomposition $f = f^d + f^c$ into positive functionals with $f^d \in A_d^*$ and $f^c \in A_c^*$.

Following EFFROS [4] we define an order ideal I in A^* to be a set of positive functionals in A^* with the property that if $f \in I$ and $0 \leqq g \leqq \lambda f$ for some $\lambda \geqq 0$, then $g \in I$. If I is a norm-closed order ideal in A^*, we define the support of I to be the complement of the largest projection p in A^{**} such that $f(p) = 0$ for all $f \in I$ [cf. 4, p. 405].

For any $a \in A^{**}$, let $a' = 1 - a$. Recall that a pure state of A is a positive f in A^* such that if $0 \leqq g \leqq \lambda f$ for some $\lambda \geqq 0$, then $g = \alpha f$ for some $\alpha \geqq 0$.

II. The Main Results

The first result characterizes those projections in A^{**} which support a weak* closed order ideal in A^*. We need only a special case for Theorem 4.

Proposition 1. *A projection p in A^{**} supports a weak* closed order ideal in A^* iff $p = \lim a_\alpha$ where $\{a_\alpha\}$ is a decreasing net of positive elements of A.*