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Quantification géométrique*

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Abstract. The usual quantization of a system with n degrees of freedom is not independent of the choice of coordinates and treats time on a special footing. In order to obtain an intrinsic quantization procedure, we have first to consider separately the phase space Φ and the set V of classical motions, which are partially set in correspondance, at each time t, by solving an initial values problem, but are nevertheless not identifiable. If the dynamical variables are defined as functions on V, Poisson's theorem defines the Poisson bracket as an internal operation on these dynamical variables; this allows us to equip V with a structure of a $2n$-dimensional symplectic manifold which follows from coordinate independent variational considerations. This symplectic structure allows us, not only to state the quantization program intrinsically (§ 2) but also to give an intrinsic realization (§ 4). With this purpose, we construct a bundle space W with basis V ("the quantizing bundle space"); the physical states are functions defined on W; to each classical dynamical variable corresponds an infinitesimal motion of W, inducing an observable of the states; whence a "quantization" fulfilling the commutation relations and the evolution equations. Since this construction uses merely the symplectic structure of V, the (local) isomorphisms of this structure can be locally lifted (along local isomorphisms of the bundle space W); the invariance groups of the system therefore acts canonically on the states, up to a phase; whence the usual projective representation of these groups, without any supplementary postulate (§ 5).

We compare the results of our geometrical quantization with the usual quantization procedure, taking as examples the harmonic oscillator and the hydrogene atom; one finds wider states spaces (due to the fact that all the dynamic variables are represented); but a general rule related to "quantal measurements" allows to restrict the spectral values e.g. of the energy to the familiar ones. The case of the free relativistic particle is treated in details; our method there provides a state on which the Poincaré group acts projectivally; by reduction of this representation, one gets spaces of complex waves functions, verifying the usual wave equations for all integer spin values. The investigation of momentum, energy and angular momentum yield natural results (§ 7).

We finally sketch the application of our method to field quantization notably of general relativity (§ 8).

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* Les principaux résultats de cet article ont été exposés pour la première fois au Séminaire de physique mathématique de la Faculté des Sciences de Marseille, en 1960—1961, et publiés en 1962 [7]; le présent exposé, notablement plus simple, est présenté sous forme inductive.

Faute de place, nous n'avons donné ici aucune démonstration; le lecteur les trouvera dans les références [8] et [9].