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On Homogenization and Scaling Limit of Some Gradient Perturbations of a Massless Free Field

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Abstract: We study the continuum scaling limit of some statistical mechanical models defined by convex Hamiltonians which are gradient perturbations of a massless free field. By proving a central limit theorem for these models, we show that their long distance behavior is identical to a new (homogenized) continuum massless free field. We shall also obtain some new bounds on the 2-point correlation functions of these models.

1. Introduction and Statement of the Main Results

In this article, we study the long distance behavior of (lattice) statistical mechanical models defined by convex Hamiltonians $H(\varphi)$ which are gradient perturbations of a massless free Gaussian. Under certain assumptions (see (H-1) and (H-2) below), we shall prove a central limit theorem for these models and show that their behavior at long distances is governed by suitable *continuum* massless Gaussians. The main idea is that these statistical mechanical models can be expressed (following Helffer and Sjöstrand [8, 12]) in terms of an infinite dimensional elliptic PDE. We develop a suitable extension of homogenization for this elliptic equation which will then yield the desired central limit theorem. Along the way, we will obtain new estimates which are pointwise versions of the Brascamp-Lieb inequalities [1]. When the Hamiltonian is a "small" perturbation of a massless free field, using the renormalization group analysis and multi-scale expansions, these models have been studied in [4, 10] and more recently in [2]. Their methods, when applicable, give more detailed information than our methods. However, our methods do not require the Hamiltonian to be a "small" perturbation of a massless free field.

In this introduction, we first define the model and then we state the main results. Then the notations for the rest of paper are introduced. A brief sketch of the proofs and organization of the rest of paper conclude the introduction.

1.1. The model. At each point x of the lattice $\mathbb{Z}^d$, there is a real random variable $\varphi(x)$ and we consider the following Hamiltonian H_m^Λ, $\Lambda = \Lambda(L) \subset \mathbb{Z}^d$ a cube of side

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