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Spectral Resonances which Become Eigenvalues

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Abstract: The stationary Schrödinger equation is $-\partial_x^2\phi + \lambda V(x)\,\phi = z\phi$ for $\phi \in \mathscr{L}^2(\mathbf{R}^+, dx)$. If the potential is bounded below, singular only at $x = 0$, negative on some compact interval and behaves like $V(x) \sim 1/x^\mu$ as $x \to \infty$ with $2 \geqq \mu > 0$, then the system admits shape resonances which continuously become eigenvalues as λ increases. Here $\lambda > 0$ and for $\mu = 2$ a sufficiently large λ is required. Exponential bounds are obtained on $\mathrm{Im}(z)$ as λ approaches a threshold. The group velocity near threshold is also estimated.

1. Introduction

We study the transition of a *spectral resonance* (s.r.) *value* to an eigenvalue which occurs at thresholds of the coupling parameter λ. A typical system is,

$$H^\lambda = -\frac{d^2}{dx^2} + \lambda V(x) \quad \text{on } \mathscr{L}^2(\mathbf{R}^+, dx), \quad \text{for } \mathbf{R}^+ \equiv (0,\infty)\ , \tag{1.1a}$$

with Dirichlet B.C. at $x = 0$ and having a shape resonance potential of the form,

$$V(x) = \begin{cases} -V_{\min}, & 0 < x < b \\ V_M, & b < x < c\ , \\ V_M(c/x)^\mu, & x > c \end{cases} \tag{1.1b}$$

where $2 \geqq \mu > 0$ and $V_{\min}, V_M$ are positive constants. The physically interesting $\mu = 2$ case requires λ sufficiently large. For $\mu > 2$ our methods break down. One serious problem is that the Agmon length of V at 0 energy is finite if $\mu > 2$. We refer the reader to [6] for a discussion which does not use shape-resonance theory.

The shape resonance problem has been studied by many authors (see [1] for an extensive list) but mostly in the non-threshold cases $-V_{\min} > V(\infty) = 0$ (see [8] for a consideration of the threshold case). Here we continue the work of [5] by studying the past-threshold case (i.e. $-V_{\min} < 0$). It is demonstrated that the appearance of eigenvalues from the bottom of the essential spectrum, in the $\mu \leqq 2$ cases, is due to the smooth transition of an s.r.value to an eigenvalue. We use an