Commun Math Phys 182, 637–660 (1996)

Communications in Mathematical Physics

On the Number of Negative Eigenvalues for a Schrödinger Operator with Magnetic Field

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Received: 29 February 1996 / Accepted: 14 May 1996

Abstract: We consider the Schrödinger operator with magnetic field

$$H = \sum_{j=1}^{n} \left(\frac{1}{i} \frac{\partial}{\partial x_j} - a_j \right)^2 + V \quad \text{in } \mathbb{R}^n \, .$$

Under certain conditions on the magnetic field $\mathbf{B} = \operatorname{curl} \mathbf{a}$, we generalize the Fefferman–Phong estimates (Bull. A. M. S. **9**, 129–206 (1983)) on the number of negative eigenvalues for $-\Delta + V$ to the operator H. Upper and lower bounds are established. Our estimates incorporate the contribution from the magnetic field. The conditions on $\mathbf{B}$ in particular are satisfied if the magnetic potentials $a_j(x)$ are polynomials.

Introduction

This paper concerns the Schrödinger operator with magnetic field:

$$H = H(\mathbf{a}, V) = \sum_{j=1}^{n} \left(\frac{1}{i} \frac{\partial}{\partial x_j} - a_j \right)^2 + V \quad \text{in } \mathbb{R}^n, \quad n \geqq 3 \, , \tag{0.1}$$

where $i = \sqrt{-1}$, $V : \mathbb{R}^n \to \mathbb{R}$ is the electric potential and $\mathbf{a} : \mathbb{R}^n \to \mathbb{R}^n$ is the magnetic potential.

Let $N(\lambda, H)$ denote the number of eigenvalues (counting multiplicity) of H smaller than λ (or in general the dimension of the spectral projection for H corresponding to the interval $(-\infty, \lambda)$). In the case $\mathbf{a}(x) \equiv \mathbf{0}$, i.e., $H = H(\mathbf{0}, V) = -\Delta + V$, a basic theorem of Cwickel, Lieb and Rosenblum states that

$$N(\lambda, -\Delta + V) \leqq c_n |\{(x, \xi) \in \mathbb{R}^n \times \mathbb{R}^n : |\xi|^2 + V(x) < \lambda\}| \, . \tag{0.2}$$

See [Si2, p. 95]. Using a sharper form of the uncertainty principle, C. Fefferman and D.H. Phong were able to refine the classical estimate (0.2). Indeed, it was shown

[1] Supported in part by NSF Grant DMS-9596266