Commun. Math. Phys. 182, 485 – 504 (1996)

Communications in Mathematical Physics

Statistical Mechanics of the 2-Dimensional Focusing Nonlinear Schrödinger Equation

David C. Brydges[1], **Gordon Slade**[2]

[1] Department of Mathematics, University of Virginia, Charlottesville, VA 22903-3199, USA. E-mail: db5d@virginia.edu

[2] Department of Mathematics and Statistics, McMaster University, Hamilton, ON, Canada L8S 4K1. E-mail: slade@mcmaster.ca

Received: 4 March 1996 / Accepted: 6 June 1996

Abstract: We study a natural construction of an invariant measure for the 2-dimensional periodic focusing nonlinear Schrödinger equation, with the critical cubic nonlinearity. We find that a phase transition occurs as the coupling constant defining the strength of the nonlinearity is increased, but that the natural construction, successful for the 1-dimensional case and for the 2-dimensional defocusing case, cannot produce an invariant measure. Our methods rely on an analysis of a statistical mechanical model closely related to the spherical model of Berlin and Kac.

1. Statement of Results

1.1. Invariant measures. The periodic nonlinear Schrödinger equation can be written

$$iu_t + \Delta u + \tfrac{1}{2}\, p\lambda |u|^{p-2} u = 0\,, \tag{1.1}$$

where $u = u(x,t)$ is complex-valued and x lies in the d-dimensional unit torus $\mathbb{T}^d$. The sign of the coupling constant λ is important for the behaviour of solutions to (1.1). In the defocusing case, corresponding to $\lambda < 0$, there are global existence results. In the focusing case, corresponding to $\lambda > 0$, for the problem on $\mathbb{R}^d$, there is a critical power $p_c = 2 + 4/d$ such that for $p < p_c$ there are global existence results while for $p > p_c$ there are finite-time blow-up results. Both behaviours can occur when $p = p_c$, depending on the value of λ and the initial value of $\|u\|_2$. These results [19] are for initial data in H_1.

This paper is primarily concerned with the focusing case for $d = 2$ and $p = p_c = 4$, and more precisely, with the construction of invariant measures for (1.1). A natural approach to the construction of an invariant measure proceeds as follows. The L^2-norm

$$\|u\|_2^2 = \int_{\mathbb{T}^d} |u(x,t)|^2\, d^d x \tag{1.2}$$

and the Hamiltonian

$$H(u) = \|\nabla u\|_2^2 - \lambda \|u\|_p^p \tag{1.3}$$