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Navier–Stokes Equations for Stochastic Particle Systems on the Lattice

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Abstract: We introduce a class of stochastic models of particles on the cubic lattice $\mathbb{Z}^d$ with velocities and study the hydrodynamical limit on the diffusive space-time scale. Assuming special initial conditions corresponding to the incompressible regime, we prove that in dimension $d \geqq 3$ there is a law of large numbers for the empirical density and the rescaled empirical velocity field. Moreover the limit fields satisfy the corresponding incompressible Navier–Stokes equations, with viscosity matrices characterized by a variational formula, formally equivalent to the Green–Kubo formula.

1. Introduction

One of the main open problems in nonequilibrium statistical physics is the derivation of the hydrodynamical equations of fluids from the microscopic Hamiltonian dynamics. The main fluid equations are the Navier–Stokes equations and the Euler equations. The Euler equations represent the conservation of macroscopic mass, energy and momentum and have an obvious hyperbolic scaling invariance

$$x \longrightarrow \alpha x, \qquad t \longrightarrow \alpha t\,. \tag{1.1}$$

The Navier–Stokes equations are more complicated and have no obvious scaling. They are given by correcting the Euler equations with viscous terms described by second order derivatives of conserved quantities such as energy, momentum and mass. In the incompressible regime, the Navier–Stokes equations become

$$\begin{aligned} &\operatorname{div} u = 0\,, \\ &\partial_t u + u \cdot \nabla u + \nabla p = \nu \Delta u \end{aligned} \tag{1.2}$$

with u the velocity field, p the pressure and $\nu > 0$ the kinematic viscosity.

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