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Splitting of the Family Index

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Abstract: We establish a general splitting formula for index bundles of families of Dirac type operators. Among the applications, our result provides a positive answer to a question of Bismut and Cheeger [BC2].

Introduction

Due to the increasing influence of the topological quantum field theory (cf. [A]), it becomes very important to study the behavior of natural analytic invariants under the splitting of manifolds. The first splitting formula, for the most fundamental invariants—the index of Dirac operators, was proved by Atiyah–Patodi–Singer [APS], in relation with their index theorem for manifolds with boundary. There is also an alternative approach surrounding the "Bojarski conjecture". For the latter see the book of Booss–Wojciechowski [BW1] for a thorough discussion.

In this paper, we treat the index bundles associated to families of Dirac type operators in this framework. Although our results should play a role in the topological field theory, our original motivation is in fact to answer a question of Bismut and Cheeger [BC2, Remark 2.3].

There are family versions of the Atiyah–Patodi–Singer index theorem due to Bismut–Cheeger [BC1, BC2] and subsequently Melrose–Piazza [MP1, MP2] among others. These formulas do imply the splitting formula for the Chern character of index bundles but fail to work on the level of K-theory. It is the purpose of this paper to establish the desired splitting formula in full generality on the K-theoretic level.

For a family of Dirac operators acting on a closed manifold Z and parametrized by B, the index bundle lives in either the K or K^1 group of the parameter space B, depending on the parity of dim Z (See [AS1 and AS2]). Now let Y be a (family of) closed hypersurface in Z separating it into two pieces Z_1, Z_2. Since Z_1 and Z_2 are now (families of) manifolds with boundary, we impose a boundary condition as in [MP1, 2] by choosing a spectral section ($cl(1)$-spectral section if $\dim Y$ is even). This is a generalization of the (now classical) Atiyah–Patodi–Singer boundary condition to the family situation. Thus let P_1, P_2 be two spectral sections (again,