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On Complex Lee and Yang Polynomials

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Abstract: We introduce a family of polynomials $P_n(z_1,\dots,z_n)$ with complex coefficients, having the same property as the real Lee and Yang polynomials: the zeros of $P_n(z,\dots,z)$ all lie on the unit circle. We deduce from this construction a class of inner functions in several complex variables.

In the book by D. Ruelle "Statistical Mechanics" [2], Ch. 5 (The Problem of Phase Transitions) begins with the study of a family of polynomials $P_n(z_1,\dots,z_n)$, introduced by Lee and Yang in 1952 (see also the article [3] by Ruelle for an update). They have the remarkable property that, for all n, the associated one-variable polynomial $P_n(z) = P_n(z,\dots,z)$ has all its roots on the unit circle. This property of the zeros of $P_n(z)$ is deduced from the following assertion on $P_n(z_1,\dots,z_n)$:

$$\text{if } P_n(z_1,\dots,z_n) = 0 \quad \text{and if } |z_1| \leqq 1,\dots,|z_{n-1}| \leqq 1, \quad \text{then } |z_n| \geqq 1\,. \tag{1}$$

Two comments are in order:

- The way (1) is proved by Lee and Yang is by means of complex analysis, and more precisely by means of "bilinear maps" of the form $z \to (az+b)/(cz+d)$; the whole proof looks quite similar to the one given by J.L. Walsh for the so-called "Walsh's contraction principle" (see J.L. Walsh [4], B. Beauzamy [1]). This is also the case for more recent proofs, such as Asano's (see [3]).
- No direct proof is known (at least to me) of the fact that the $P_n(z)$'s have their roots on the unit circle (except in a special case, see below). Apparently, one "has" to go through the "decoupling procedure" $P_n(z_1,\dots,z_n)$ in order to ensure this.

We say that we have a decoupling procedure if we can replace a polynomial $P_n(z)$ in one variable by a polynomial $P(z_1,\dots,z_n)$, linear in each z_j (not necessarily symmetric), with the property that $P(z,\dots,z) = P(z)$. The author gave in [1]

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