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Gravity Coupled with Matter and the Foundation of Non-commutative Geometry

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Abstract: We first exhibit in the commutative case the simple algebraic relations between the algebra of functions on a manifold and its infinitesimal length element ds. Its unitary representations correspond to Riemannian metrics and Spin structure while ds is the Dirac propagator $ds = \times\!\!-\!\!\times = D^{-1}$, where D is the Dirac operator. We extend these simple relations to the non-commutative case using Tomita's involution J. We then write a spectral action, the trace of a function of the length element, which when applied to the non-commutative geometry of the Standard Model will be shown ([CC]) to give the SM Lagrangian coupled to gravity. The internal fluctuations of the non-commutative geometry are trivial in the commutative case but yield the full bosonic sector of SM with all correct quantum numbers in this slightly non-commutative case. The group of local gauge transformations appears spontaneously as a normal subgroup of the diffeomorphism group.

Riemann's concept of a geometric space is based on the notion of a manifold M whose points $x \in M$ are locally labelled by a finite number of real coordinates $x^\mu \in \mathbb{R}$. The metric data is given by the infinitesimal length element,

$$ds^2 = g_{\mu\nu}\, dx^\mu\, dx^\nu, \tag{1}$$

which allows to measure distances between two points x, y as the infimum of the length of arcs $x(t)$ from x to y,

$$d(x,y) = \operatorname{Inf} \int_x^y ds\,. \tag{2}$$

In this paper we shall build our notion of geometry, in a very similar but somehow dual manner, on the pair $(\mathscr{A}, ds)$ of the algebra $\mathscr{A}$ of coordinates and the infinitesimal length element ds. To start we only consider ds as a symbol, which together with $\mathscr{A}$ generates an algebra $(\mathscr{A}, ds)$. The length element ds does not commute with the coordinates, i.e. with the functions f on our space, $f \in \mathscr{A}$. But it does satisfy non-trivial relations. Thus in the simplest case where $\mathscr{A}$ is commutative we shall have,

$$[[f, ds^{-1}], g] = 0 \quad \forall f, g \in \mathscr{A}\,. \tag{3}$$