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Smoothness and Non-Smoothness of the Fundamental Solution of Time Dependent Schrödinger Equations

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Abstract: The fundamental solution $E(t,s,x,y)$ of time dependent Schrödinger equations $i\partial u/\partial t = -(1/2)\triangle u + V(t,x)u$ is studied. It is shown that

- $E(t,s,x,y)$ is smooth and bounded for $t \neq s$ if the potential is sub-quadratic in the sense that $V(t,x) = o(|x|^2)$ at infinity;
- in one dimension, if $V(t,x) = V(x)$ is time independent and super-quadratic in the sense that $V(x) \geqq C(1+|x|)^{2+\varepsilon}$ at infinity, $C > 0$ and $\varepsilon > 0$, then $E(t,s,x,y)$ is nowhere C^1.

The result is explained in terms of the limiting behavior as the energy tends to infinity of the corresponding classical particle.

1. Introduction

We consider the time dependent Schrödinger equation with a real potential $V(t,x)$:

$$i\partial u/\partial t = -(1/2)\triangle u + V(t,x)u, \quad (t,x) \in \mathbf{R}^1 \times \mathbf{R}^m . \tag{1.1}$$

The equation generates a unique unitary propagator $\{U(t,s) : -\infty < t,s < \infty\}$ in $L^2(\mathbf{R}^m)$ under the conditions to be imposed below and $u(t,x) = (U(t,s)\phi)(x)$ represents a unique solution of (1.1) which satisfies the initial condition $u(s,x) = \phi(x) \in L^2(\mathbf{R}^m)$. Standard arguments show $U(t,s)$ is a two parameter family of strongly continuous unitary operators satisfying the semi-group properties: $U(t,t) = 1$ and $U(t,s)U(s,r) = U(t,r)$. We denote by $E(t,s,x,y)$ the distribution kernel of $U(t,s)$: $E = E(t,s,x,y)$ is the fundamental solution of Eq. (1.1), or FDS for short. In this paper, we show that

1. $E(t,s,x,y)$ is smooth and bounded with respect to (x,y) for any $t \neq s$, provided V is "sub-quadratic" in the sense that for all $|\alpha| = 2$, $\lim_{|x|\to\infty} |\partial_x^\alpha V(t,x)| = 0$ uniformly with respect to $t \in \mathbf{R}^1$;
2. in one dimension, if $V(t,x) = V(x)$ is time independent and "super-quadratic" in the sense that $V(x) \geqq C(1+|x|)^{2+\varepsilon}$ at infinity, $C > 0$ and $\varepsilon > 0$, then $E(t,s,x,y)$ is nowhere C^1.