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Logarithmic Sobolev Inequality for Lattice Gases with Mixing Conditions

Horng-Tzer Yau★

Courant Institute, New York University, New York, NY 10012, USA.
E-mail: yau@math.nyu edu

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Abstract: Let $\mu^{gc}_{\Lambda_L,\lambda}$ denote the grand canonical Gibbs measure of a lattice gas in a cube of size L with the chemical potential λ and a fixed boundary condition. Let $\mu^{c}_{\Lambda_L,n}$ be the corresponding canonical measure defined by conditioning $\mu^{gc}_{\Lambda_L,\lambda}$ on $\sum_{x\in\Lambda}\eta_x = n$. Consider the lattice gas dynamics for which each particle performs random walk with rates depending on near-by particles. The rates are chosen such that, for every n and L fixed, $\mu^{c}_{\Lambda_L,n}$ is a reversible measure. Suppose that the Dobrushin–Shlosman mixing conditions holds for $\mu^{gc}_{L,\lambda}$ for *all* chemical potentials $\lambda \in \mathbb{R}$. We prove that $\int f \log f d\mu^{c}_{\Lambda_L,n} \leqq$ const. $L^2 D(\sqrt{f})$ for any probability density f with respect to $\mu^{c}_{\Lambda_L,n}$; here the constant is independent of n or L and D denotes the Dirichlet form of the dynamics. The dependence on L is optimal.

I. Introduction

Suppose that $\mathscr{L}$ is the generator of a dynamics and that μ is an invariant measure. The Dirichlet form of a function g is defined by

$$D(g) = -\int g\mathscr{L}g\, d\mu\,.$$

As only the symmetric part of the generator enters in this definition, we may as well assume that the dynamics is reversible, i.e., $\mathscr{L}$ is symmetric with respect to μ. A logarithmic Sobolev inequality for this system states that the entropy of a probability density f with respect to μ can be bounded by a constant multiple of the Dirichlet form, namely,

$$\int f \log f d\mu \leqq \kappa D(\sqrt{f})\,.$$

It is well-known that the logarithmic Sobolev inequality is equivalent to the hypercontractivity of the semigroup and thus it provides certain information on the

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