Toda Lattice Hierarchy and Generalized String Equations

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Abstract: String equations of the pth generalized Kontsevich model and the compactified $c = 1$ string theory are re-examined in the language of the Toda lattice hierarchy. As opposed to a hypothesis postulated in the literature, the generalized Kontsevich model at $p = -1$ does not coincide with the $c = 1$ string theory at self-dual radius. A broader family of solutions of the Toda lattice hierarchy including these models is constructed, and shown to satisfy generalized string equations. The status of a variety of $c \leqq 1$ string models is discussed in this new framework.

1. Introduction

The so-called "string equations" play a key role in various applications of integrable hierarchies to low dimensional string theories. The most fundamental integrable hierarchy in this context is the KP hierarchy [31] that provides a universal framework for dealing with many KdV-type hierarchies. String equations for "(p, q) models" of two-dimensional quantum gravity can be treated in a unified manner in this language. In contrast, the status of the Toda lattice hierarchy [37], which is another universal integrable hierarchy, had remained relatively obscure until rather recent years. The Toda lattice hierarchy was pointed out to be an integrable structure of the one- and multi-matrix models [13], but these matrix models (matrix integrals) were only considered as an intermediate step towards the continuous (double scaling) limit to two-dimensional quantum gravity.

In the last few years, the Toda lattice hierarchy has come to be studied from renewed points of view, such as $c = 1$ strings [7, 23, 27, 28], two-dimensional topological strings [14, 16, 10, 34, 5], the topological CP^1 sigma model and its variations related to affine Coxeter groups [11, 18, 9]. As opposed to the (p, q) models in the KP hierarchy, these are related to string theories with a true continuous target space. Our goal in this paper is to elucidate the structure of those string equations, in particular, those of $c = 1$ strings in a more general framework.

It will be instructive to recall the relationship between the (p, q) models and the KP hierarchy. String equations of these models were first discovered in the form of