Commun. Math. Phys 180, 653–670 (1996)

Communications in
Mathematical
Physics

Graph Invariants of Vassiliev Type and Application to 4D Quantum Gravity

Nobuharu Hayashi

Institute of Physics, University of Tokyo, Komaba, Tokyo 153, Japan†
E-mail: hayashin@kekvax.kek.jp

Received: 3 April 1995 / Accepted: 26 January 1996

Abstract: We consider graph invariants of Vassiliev type extended by the quantum group link invariants. When they are expanded by x where $q = e^x$, the expansion coefficients are known as the Vassiliev invariants of finite type. In the present paper, we define tangle operators of graphs given by a functor from a category of colored and oriented graphs embedded into a 3-space to a category of representations of the quasi-triangular ribbon Hopf algebra extended by $U_q(sl(2, C))$, which are subject to a quantum group analog of the spinor identity. In terms of them, we obtain the graph invariants of Vassiliev type expressed to be identified with Chern–Simons vacuum expectation values of Wilson loops including intersection points. We also consider the 4d canonical quantum gravity of Ashtekar. It is verified that the graph invariants of Vassiliev type satisfy constraints of the quantum gravity in the loop space representation of Rovelli and Smolin.

1. Introduction

The concept of Vassiliev invariants was introduced in the theory of knot spaces [30]. Let $\mathscr{M}$ be a space of all smooth maps $S^1 \to S^3$ running through a base point with a fixed tangent vector. The knot space is given by $\mathscr{M} \backslash \Sigma$, where Σ is called the *discriminant*, i.e., a set of all singular maps which have multiple points or vanishing tangent vectors. Equivalence classes of knot embeddings by the ambient isotopy of S^3 are in one-to-one correspondence with connected components of $\mathscr{M} \backslash \Sigma$. Each connected component of the knot space is separated by walls which constitute the discriminant Σ.

Vassiliev introduced a system of subgroups of $\tilde{H}^0(\mathscr{M} \backslash \Sigma)$ based on the Alexander duality theorem applied to a space of polynomial maps which approximates $\mathscr{M}$: $\tilde{H}^0(\mathscr{M} \backslash \Sigma) \supset \cdots \supset F_j \supset F_{j-1} \supset \cdots \supset F_1 = 0$. Elements in a quotient group F_j/F_{j-1} are called the Vassiliev invariants of order j. Every element in F_j/F_{j-1} vanishes whenever there are more than j transverse double points. The Vassiliev

† This is not the author's present address.