Commun Math. Phys 180, 233–263 (1996)

Communications in
Mathematical
Physics

A Matrix Integral Solution to $[P, Q] = P$ and Matrix Laplace Transforms

M. Adler[1,*], **A. Morozov**[2,**], **T. Shiota**[3,***], **P. van Moerbeke**[1,4,†]

[1] Department of Mathematics, Brandeis University, Waltham, Mass 02254, USA
E-mail: adler@math.brandeis edu, vanmoerbeke@math brandeis edu
[2] ITEP, Moscow, Russia E-mail: morozov@vitep5.itep ru
[3] Department of Mathematics, Kyoto University, Kyoto 606–01, Japan
E-mail: shiota@kusm kyoto-u ac.jp
[4] Department of Mathematics, Universite de Louvain, 1348 Louvain-la-Neuve, Belgium
E-mail: vanmoerbeke@geom ucl.ac.be

Received: 5 December 1994 / Accepted: 10 February 1996

Abstract: In this paper we solve the following problems: (i) find two differential operators P and Q satisfying $[P, Q] = P$, where P flows according to the KP hierarchy $\partial P/\partial t_n = [(P^{n/p})_+, P]$, with $p := \operatorname{ord} P \geq 2$; (ii) find a matrix integral representation for the associated τ-function. First we construct an infinite dimensional space $\mathscr{W} = \operatorname{span}_{\mathbb{C}}\{\psi_0(z), \psi_1(z), \ldots\}$ of functions of $z \in \mathbb{C}$ invariant under the action of two operators, multiplication by z^p and $A_c := z\,\partial/\partial z - z + c$. This requirement is satisfied, for arbitrary p, if ψ_0 is a certain function generalizing the classical Hänkel function (for $p = 2$); our representation of the generalized Hänkel function as a double Laplace transform of a simple function, which was unknown even for the $p = 2$ case, enables us to represent the τ-function associated with the KP time evolution of the space $\mathscr{W}$ as a "double matrix Laplace transform" in two different ways. One representation involves an integration over the space of matrices whose spectrum belongs to a wedge-shaped contour $\gamma := \gamma^+ + \gamma^- \subset \mathbb{C}$ defined by $\gamma^\pm = \mathbb{R}_+ e^{\pm \pi i/p}$. The new integrals above relate to matrix Laplace transforms, in contrast with matrix Fourier transforms, which generalize the Kontsevich integrals and solve the operator equation $[P, Q] = 1$.

Table of Contents

* The support of a National Science Foundation grant #DMS-95-4-51179 is gratefully acknowledged
** The hospitality of the Volterra Center at Brandeis University is gratefully acknowledged
*** The hospitality of the University of Louvain and Brandeis University is gratefully acknowledged.
† The support of a National Science Foundation grant #DMS-95-4-51179, a Nato, an FNRS and a Francqui Foundation grant is gratefully acknowledged
Correspondence to: P van Moerbeke