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Spiders for Rank 2 Lie Algebras

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Abstract: A spider is an axiomatization of the representation theory of a group, quantum group, Lie algebra, or other group or group-like object. It is also known as a spherical category, or a strict, monoidal category with a few extra properties, or by several other names. A recently useful point of view, developed by other authors, of the representation theory of sl(2) has been to present it as a spider by generators and relations. That is, one has an algebraic spider, defined by invariants of linear representations, and one identifies it as isomorphic to a combinatorial spider, given by generators and relations. We generalize this approach to the rank 2 simple Lie algebras, namely A_2, B_2, and G_2. Our combinatorial rank 2 spiders yield bases for invariant spaces which are probably related to Lusztig's canonical bases, and they are useful for computing quantities such as generalized $6j$-symbols and quantum link invariants. Their definition originates in definitions of the rank 2 quantum link invariants that were discovered independently by the author and Francois Jaeger.

1. Introduction

One of the problems of classical invariant theory is to characterize, for all n-tuples $V_1, \ldots, V_n$ of finite-dimensional, irreducible representations over $\mathbb{C}$ of a compact group G or simple Lie algebra $\mathfrak{g}$, the vector space of multilinear functions

$$f : V_1 \times V_2 \times \cdots \times V_n \to \mathbb{C}$$

which are invariant under the action of G or $\mathfrak{g}$. In more modern terminology, the problem is to characterize the dual vector space of invariant tensors

$$\mathrm{Inv}(V_1 \otimes V_2 \otimes \cdots \otimes V_n)\,,$$

or just $\mathrm{Inv}(V)$ if V is a tensor product of irreducibles. (Also, instead of working over $\mathbb{C}$, one might work over some other field $\mathbb{F}$ of characteristic 0.) Of course, for a simple Lie algebra $\mathfrak{g}$, the dimension of such a vector space is given by Cartan–Weyl

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