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# On the Deformability of Heisenberg Algebras

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**Abstract:** Based on the vanishing of the second Hochschild cohomology group of the Weyl algebra it is shown that differential algebras coming from quantum groups do not provide a non-trivial deformation of quantum mechanics. For the case of a $q$-oscillator there exists a deforming map to the classical algebra. It is shown that the differential calculus on quantum planes with involution, i.e., if one works in position-momentum realization, can be mapped on a $q$-difference calculus on a commutative real space. Although this calculus leads to an interesting discretization it is proved that it can be realized by generators of the undeformed algebra and does not possess a proper group of global transformations.

## 1. Introduction

It is known that the deformation of an algebra, either of Lie or associative type, is connected to its (Chevalley or Hochschild) cohomology [15]. More precisely, for an algebra $\mathbf{g}$ the second cohomology group $H^2(\mathbf{g},\mathbf{g})$ contains the information if a non-trivial deformation of it exists or not. In particular, if $H^2(\mathbf{g},\mathbf{g}) = 0$, then there exists no non-trivial deformation of $\mathbf{g}$.

This result can readily be applied to the case of quantum groups [8, 17]. Here one takes for example a finite-dimensional semisimple Lie algebra $\mathbf{g}$ and addresses the question of existence of deformations of its enveloping algebra $\mathscr{U}(\mathbf{g})$. It is well known that we have non-trivial deformations denoted by $\mathscr{U}_h(\mathbf{g})$ as long as one considers $\mathscr{U}_h(\mathbf{g})$ as being a Hopf algebra or at least a bialgebra. The non-triviality of this deformation comes from the fact that $H^2(\mathscr{U}(\mathbf{g}),\mathscr{U}(\mathbf{g}))_{\text{bialgebra}} \simeq \Lambda^2(\mathbf{g}) \neq 0$ [9, 18, Ch.18], where $\Lambda(\mathbf{g})$ denotes the exterior algebra.

In contrast if one would consider only the algebra part of $\mathscr{U}(\mathbf{g})$ the classical Whitehead lemma applies in this case. That lemma states that for a finite-dimensional semisimple Lie algebra $\mathbf{g}$ and a finite-dimensional left-$\mathbf{g}$-module $M$ it holds that:

$$H^1(\mathbf{g},M) = H^2(\mathbf{g},M) = 0 \,. \qquad (1)$$

[1] We take for the deformation parameter $q = e^h > 1$ throughout this paper.