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A Local-to-Global Singularity Theorem for Quantum Field Theory on Curved Space-Time

Marek J. Radzikowski[1,*]
(with an Appendix by Rainer Verch[2]**)**

[1] Department of Mathematics, Texas A&M University, College Station, TX 77843, USA, and Department of Mathematics, University of York, Heslington, York YO1 5DD, UK
[2] II Institut für Theoretische Physik, Universität Hamburg, D - 22761 Hamburg, Germany

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To a special friend, who saved my life when I was younger, without whom I could not have written this paper.

Abstract: We prove that if a reference two-point distribution of positive type on a time orientable curved space-time (CST) satisfies a certain condition on its wave front set (the "class $\mathscr{P}_{M,g}$ condition") and if any other two-point distribution (i) is of positive type, (ii) has the same antisymmetric part as the reference modulo smooth function and (iii) has the same local singularity structure, then it has the same global singularity structure. In the proof we use a smoothing, positivity-preserving pseudo-differential operator the support of whose symbol is restricted to a certain conic region which depends on the wave front set of the reference state. This local-to-global theorem, together with results published elsewhere, leads to a verification of a conjecture by Kay that for quasi-free states of the Klein-Gordon quantum field on a globally hyperbolic CST, the local Hadamard condition implies the global Hadamard condition. A counterexample to the local-to-global theorem on a strip in Minkowski space is given when the class $\mathscr{P}_{M,g}$ condition is not assumed.

1. Introduction

In the quantum field theory (QFT) of a Klein-Gordon scalar field on a globally hyperbolic curved space-time (CST) [2, 11], the Hadamard condition [18, 5] is believed to be a "physically necessary" condition on the two-point distribution of a quasi-free or more general state [13, 12, 28, 26]. Some reasons for this belief arose from investigations into the point-splitting renormalization technique used in defining observables quadratic in the field operators on such space-times. It was discovered that the Hadamard condition is sufficient for point-splitting renormalization to yield a stress-energy tensor $T_{\mu\nu}(x)$ that satisfies a set of properties encapsulating what is meant by "physically meaningful." These are called the *Wald axioms* [43, 44]. The *local Hadamard condition* (LH) specifies the asymptotic behavior of the two-point distribution $\omega_2(x_1, x_2)$ for x_1 close to x_2 to be

* Present address: Department of Mathematics, University of Toronto, Toronto, Ontario M5S 3G3, Canada, E-mail: marek@math.toronto.edu