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Proofs of Two Conjectures Related to the Thermodynamic Bethe Ansatz

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Abstract: We prove that the solution to a pair of nonlinear integral equations arising in the thermodynamic Bethe Ansatz can be expressed in terms of the resolvent kernel of the linear integral operator with kernel

$$\frac{e^{-(u(\theta)+u(\theta'))}}{\cosh\frac{\theta-\theta'}{2}}\,.$$

I. Introduction

Thermodynamic Bethe Ansatz techniques were introduced in the pioneering analysis of Yang and Yang [11] of the thermodyamics of a nonrelativistic, one-dimensional Bose gas with delta function interaction. Later this method was extended to a relativistic system with a factorizable S-matrix to give an exact expression for the ground state energy of this system on a cylindrical space of circumference R [5, 12]. This was done by relating the ground state energy to the free energy of the same system on an infinite line at temperature $T = 1/R$. In all cases one expresses the various quantities of interest in terms of "excitation energies" $\varepsilon_a(\theta)$ which are solutions of nonlinear integral equations of the form

$$\varepsilon_a(\theta) = u_a(\theta) - \sum_b \int \phi_{ab}(\theta-\theta')\log(1+z_b\,e^{-\varepsilon_b(\theta')})\,\frac{d\theta'}{2\pi} \quad (a = 1,2,\dots)\,,$$

where $\phi_{ab}(\theta)$ are expressible in terms of the 2-body S-matrix, z_a are activities, and for relativistic systems $u_a(\theta) = m_aR\cosh\theta$. These nonlinear integral equations are the so-called *thermodynamic Bethe Ansatz* (TBA) *equations*. Solving the TBA equations is another matter. The methods used are either numerical or perturbative and there are, as far as the authors are aware, no known explicit solutions to the TBA equations.

It thus came as a surprise when Cecotti et al. [2] (see also [3]), in their analysis of certain $N = 2$ supersymmetric theories [1], discovered that a certain quantity