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Fredholm Determinants and the mKdV/ Sinh–Gordon Hierarchies

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Abstract: For a particular class of integral operators K we show that the quantity

$$\phi := \log\det(I+K) - \log\det(I-K)$$

satisfies both the integrated mKdV hierarchy and the Sinh–Gordon hierarchy. This proves a conjecture of Zamolodchikov.

I. Introduction

In recent years it has become apparent that there is a fundamental connection between certain Fredholm determinants and total systems of differential equations. This connection first appeared in work on the scaling limit of the 2-point correlation function in the 2D Ising model [7, 15] and the subsequent generalization to n-point correlations and holonomic quantum fields [12]. In applications the Fredholm determinants are either correlation functions or closely related to correlation functions in various statistical mechanical or quantum field-theoretic models. In the simplest of cases the differential equations are one of the Painlevé equations. Some, but by no means a complete set of, references to these further developments are [2–5, 13, 14, 16] The review paper [6] can be consulted for more examples of this connection.

In recent work by the present authors on random matrices, techniques were developed that gave simple proofs of the connection between a large class of Fredholm determinants and differential equations [13, 14]. In this paper we show how the philosophy of [3, 5, 13, 14] can be applied to study Fredholm determinants which are associated with operators K having kernel of the form

$$K(x,y) = \frac{E(x)\,E(y)}{x+y}\,,$$

where

$$E(x) = e(x)\,\exp\left(\sum \tfrac{1}{2}\,t_k\,x^k\right)\,.$$