Fredholm Determinants and the mKdV/Sinh–Gordon Hierarchies

Craig A. Tracy, Harold Widom

1 Department of Mathematics and Institute of Theoretical Dynamics, University of California, Davis, CA 95616, USA
2 Department of Mathematics, University of California, Santa Cruz, CA 95064, USA

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Abstract: For a particular class of integral operators $K$ we show that the quantity

$$\phi := \log \det (I + K) - \log \det (I - K)$$

satisfies both the integrated mKdV hierarchy and the Sinh–Gordon hierarchy. This proves a conjecture of Zamolodchikov.

I. Introduction

In recent years it has become apparent that there is a fundamental connection between certain Fredholm determinants and total systems of differential equations. This connection first appeared in work on the scaling limit of the 2-point correlation function in the 2D Ising model [7, 15] and the subsequent generalization to $n$-point correlations and holonomic quantum fields [12]. In applications the Fredholm determinants are either correlation functions or closely related to correlation functions in various statistical mechanical or quantum field-theoretic models. In the simplest of cases the differential equations are one of the Painlevé equations. Some, but by no means a complete set of, references to these further developments are [2–5, 13, 14, 16] The review paper [6] can be consulted for more examples of this connection.

In recent work by the present authors on random matrices, techniques were developed that gave simple proofs of the connection between a large class of Fredholm determinants and differential equations [13, 14]. In this paper we show how the philosophy of [3, 5, 13, 14] can be applied to study Fredholm determinants which are associated with operators $K$ having kernel of the form

$$K(x, y) = \frac{E(x) E(y)}{x + y},$$

where

$$E(x) = e(x) \exp \left( \sum \frac{1}{2} t_k x^k \right).$$