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Iso-Spectral Deformations of General Matrix and Their Reductions on Lie Algebras

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Abstract: We study an iso-spectral deformation of the general matrix which is a natural generalization of the nonperiodic Toda lattice equation. This deformation is equivalent to the Cholesky flow, a continuous version of the Cholesky algorithm, introduced by Watkins. We prove the integrability of the deformation and give an explicit formula for the solution to the initial value problem. The formula is obtained by generalizing the orthogonalization procedure of Szegö. Using the formula, the solution to the LU matrix factorization can be constructed explicitly. Based on the root spaces for simple Lie algebras, we consider several reductions of the equation. This leads to generalized Toda equations related to other classical semi-simple Lie algebras which include the integrable systems studied by Bogoyavlensky and Kostant. We show these systems can be solved explicitly in a unified way. The behaviors of the solutions are also studied. Generically, there are two types of solutions, having either sorting property or blowing up to infinity in finite time.

1. Introduction

In this paper we consider an iso-spectral deformation of an arbitrary diagonalizable matrix $L \in \mathfrak{M}(N, \mathbb{R})$. With the deformation parameter $t \in \mathbb{R}$, this is defined by

$$\frac{d}{dt}L = [P, L] , \tag{1.1}$$

where P is the generating matrix of the deformation and is given by a projection of L,

$$P = \Pi(L) := (L)_{>0} - (L)_{<0} . \tag{1.2}$$

Here $(L)_{>0\ (<0)}$ denotes the strictly upper (lower) triangular part of L. In terms of the standard basis of $\mathfrak{M}(N, \mathbb{R})$, i.e.,

$$E_{ij} = (\delta_{ik}\delta_{jl})_{1 \leqq k, l \leqq N} , \tag{1.3}$$

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