Commun. Math. Phys. 178, 733–746 (1996)

Communications in Mathematical Physics

The Spectrum of Relativistic One-Electron Atoms According to Bethe and Salpeter

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Received: 23 May 1995/Accepted: 20 October 1995

Abstract: Bethe and Salpeter introduced a relativistic equation – different from the Bethe–Salpeter equation – which describes relativistic multi-particle systems. Here we will begin some basic work concerning its mathematical structure. In particular we show self-adjointness of the one-particle operator which will be a consequence of a sharp Sobolev type inequality yielding semi-boundedness of the corresponding sesquilinear form. Moreover we locate the essential spectrum of the operator and show the absence of singular continuous spectrum.

1. Introduction

It is well known that the extension of the Dirac equation to multi-particle systems in analogy with the multi-particle Schrödinger equation is problematic. Already the operator describing two non-interacting electrons in the electric field of a nucleus can be easily seen to have the whole real line as spectrum. The situation does not improve when the interaction between the electrons is taken into account. This trivial but important remark was seemingly made rather late (Brown and Ravenhall [2]) and is known in the physics literature as continuum dissolution.

Bethe and Salpeter [1] proposed an equation that overcomes this difficulty by projecting to the electron subspace only. Note that the Dirac equation really describes two different particles, namely electrons and positrons. Although their intention is clearly to treat the multi-particle problem, it is mathematically interesting to discuss the one-particle operator first, since its properties are basic for the N-body situation. The Hamiltonian B of Bethe and Salpeter – we will henceforth use the term Bethe–Salpeter operator – for an electron of charge $-e$ in the magnetic vector potential $\mathfrak{A}$ and the electric potential φ is

$$B = \Lambda_+ \left(c\boldsymbol{\alpha}\cdot\left(\frac{\hbar}{i}\mathrm{grad} + e\mathfrak{A}\right) + mc^2\beta - e\varphi\right)\Lambda_+ , \qquad (1)$$