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Diffusive Stability of Spatial Periodic Solutions of the Swift–Hohenberg Equation

Guido Schneider
IfAM, Universität Hannover, Welfengarten 1, D-30167 Hannover, Germany

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Abstract: We are interested in the nonlinear stability of the Eckhaus-stable equilibria of the Swift–Hohenberg equation on the infinite line with respect to small integrable perturbations. The difficulty in showing stability for these stationary solutions comes from the fact that their linearizations possess continuous spectrum up to zero. The nonlinear stability problem is treated with renormalization theory which allows us to show that the nonlinear terms are irrelevant, i.e. that the fully nonlinear problem behaves asymptotically as the linearized one which is under a diffusive regime.

1. Introduction

We consider evolutionary problems on the infinite line. Here we are interested in the nonlinear stability of the bifurcating, spatial periodic, stationary solutions u_0 of the Swift–Hohenberg equation with respect to small integrable, non-periodic perturbations. Under stability we understand the following generalization of the usual stability definition ($\mathscr{B}_1 = \mathscr{B}_2$).

Definition 1. *Let $\mathscr{B}_1, \mathscr{B}_2$ be Banach spaces and let S_t be an evolution operator. A fixed point $u_0 = S_t u_0$ is called $(\mathscr{B}_1, \mathscr{B}_2)$-stable under S_t if the following holds: For all $\mu > 0$ there exists a $\delta > 0$ such that from $\|v - u_0\|_{\mathscr{B}_1} < \delta$ follows that $\|S_t v - u_0\|_{\mathscr{B}_2} < \mu$ for all $t \geqq 0$. The point u_0 is called asymptotically $(\mathscr{B}_1, \mathscr{B}_2)$-stable if additionally $\lim_{t\to\infty} S_t v = u_0$ in $\mathscr{B}_2$ holds.*

The first step in stability proofs is the examination of the spectrum of the linearization around u_0. As always for translationally invariant problems the linearization around the nontrivial ground state u_0 possesses $\partial_x u_0$ as eigenvector to the eigenvalue zero. Since zero is the maximal eigenvalue and no Jordan-block appears the stability question has to be answered through the nonlinear terms. In cases of an appearing spectral gap between zero and the negative eigenvalues center manifold theory applies to prove nonlinear stability. The center manifold is usually equal to the family of stationary solutions $\{u_0(s + \cdot\,) \mid s \in \mathbb{R}\}$. Thus, asymptotic phase theory [He81] predicts that every solution nearby the center manifold converges to