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# Volumes of Restricted Minkowski Sums and the Free Analogue of the Entropy Power Inequality

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**Abstract:** In noncommutative probability theory independence can be based on free products instead of tensor products. This yields a highly noncommutative theory: free probability theory (for an introduction see [9]). The analogue of entropy in the free context was introduced by the second named author in [8]. Here we show that Shannon's entropy power inequality ([6, 1]) has an analogue for the free entropy $\chi(X)$ (Theorem 2.1).

The free entropy, consistent with Boltzmann's formula $S = k \log W$, was defined via volumes of matricial microstates. Proving the free entropy power inequality naturally becomes a geometric question.

Restricting the Minkowski sum of two sets means to specify the set of pairs of points which will be added. The relevant inequality, which holds when the set of addable points is sufficiently large, differs from the Brunn–Minkowski inequality by having the exponent $1/n$ replaced by $2/n$. Its proof uses the rearrangement inequality of Brascamp–Lieb–Lüttinger ([2]). Besides the free entropy power inequality, note that the inequality for restricted Minkowski sums may also underlie the classical Shannon entropy power inequality (see 3.2 below).

## 1. The Inequality for Restricted Minkowski Sums

If $A, B \subset \mathbb{R}^n$ (or any vector space), the Minkowski sum of $A$ and $B$ is defined by

$$A + B = \{x + y : (x, y) \in A \times B\} .$$

An important property of the Minkowski sum in $\mathbb{R}^n$ is the Brunn–Minkowski inequality ([4, 5])

$$\lambda(A + B)^{1/n} \geqq \lambda(A)^{1/n} + \lambda(B)^{1/n} ,$$

where $\lambda$ denotes $n$-dimensional Lebesgue measures. We introduce a modified concept of a sum.

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