Commun. Math. Phys. 178, 265–280 (1996)

Communications in
Mathematical
Physics

Growth and Oscillations of Solutions of Nonlinear Schrödinger Equation

Sergei B. Kuksin
Steklov Mathematical Institute, Vavilova St. 42, 117966 Moscow, Russia

Received: 20 May 1995 / Accepted: 29 August 1995

To the memory of Natasha and Sergei Kozlov

Abstract: We study the nonlinear Schrödinger equation in an n-cube, $n = 1,2,3$, under Dirichlet boundary conditions, treating it as a dynamical system in a function space formed by sufficiently smooth functions of x. We show that this space contains a distinguished small subset $\mathfrak{A}$ which is a recursion subset for the dynamical system and describe the dynamics of the equation in terms of the trajectory's recurrence to $\mathfrak{A}$. We use this description to estimate from below the space- and time-space oscillations of solutions in terms of a quantity, similar to the Reynolds number of classical hydrodynamics.

Introduction

We consider the nonlinear Schrödinger equation

$$-i\dot{u} = \delta(-\Delta u + V(x)u) + |u|^2 u, \quad u = u(t,x), \ \delta > 0\,, \tag{1}$$

with the space-variable x in the n-cube $K^n = \{0 \leqq x_j \leqq \pi\}$, $n = 1,2,3$, under Dirichlet boundary conditions

$$u|_{\partial K^n} = 0\,. \tag{2}$$

We study the problem (1), (2) as a dynamical system in a function space Z formed by sufficiently smooth complex functions $u(x)$,

$$Z \subset C^m(K^n;\mathbb{C}), \quad m \geqq 3\,, \tag{3}$$

which vanish at ∂K^n. That is, given $u_0 \in Z$ we interpret the solution $u(t,x)$ of (1), (2) with $u(0,x) = u_0(x)$ as a curve $u(t) \in Z$ and study the trajectories $u(t)$ as well as the flow-maps $S^t : Z \to Z$, $u_0 \mapsto u(t)$.

The problem (1), (2) is well-known to be Hamiltonian with the Hamiltonian $\mathscr{H}$,

$$\mathscr{H}(u(x)) = \int \left(\frac{\delta}{2}\,|\nabla u(x)|^2 + \frac{\delta}{2}\,V(x)\,|u(x)|^2 + \frac{1}{4}|u(x)|^4 \right) dx \Big/ (2\pi)^n\,, \tag{4}$$