Čech Cocycles for Characteristic Classes

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Abstract: We give general formulae for explicit Čech cocycles representing characteristic classes of real and complex vector bundles, as well as for cocycles representing Chern–Simons classes of bundles with arbitrary connections. Our formulae involve integrating differential forms over moving simplices inside homogeneous spaces. An important feature of our cocycles is that they take integer values (as opposed to real or rational values). We find in particular a formula for the instanton number of a connection over a closed four-manifold with arbitrary structure group. For flat connections, our formulae recover and generalize those of Cheeger and Simons. The methods of this paper apply also to the purely geometric construction of the Quillen line bundle with its metric.

A vector bundle $E \to M$ has characteristic classes (Chern, Pontryagin and Euler classes) in integral cohomology groups $H^p(M, \mathbb{Z})$. The Chern–Weil theory gives differential forms which represent the corresponding classes in the real cohomology groups $H^p(M, \mathbb{R})$. Gelfand posed the problem of finding a combinatorial formula for integer-valued singular cocycles representing the Pontryagin classes. This is considerably more difficult than finding a real-valued cocycle, which can easily be done using a partition of unity [5].

An explicit formula for a singular cocycle representing 24 times the first Pontryagin class $p_1(M)$ of a smooth manifold M was found by I. Gelfand, Gabrielov and Losik [16] and by MacPherson [20]. This formula, which involves the dilogarithm function, has had considerable influence in algebraic topology and in algebraic K-theory. More recently, Gelfand and MacPherson [17] gave a formula for a *rational* simplicial cocycle representing any Pontryagin class of a smooth polyhedral manifold.

In this paper we work with Čech cohomology instead of singular cohomology. We give a direct and explicit construction of *all* the *integer-valued* Čech cocycles