Commun. Math. Phys. 177, 507 – 528 (1996)

Communications in
Mathematical
Physics

Quantum Ergodicity of C^* Dynamical Systems

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Received: 1 April 1995

Abstract: We define a notion of quantum, or non-commutative, ergodicity for a class of C^*-dynamical systems $(\mathscr{A}, G, \alpha)$ which we call *quantized GNS systems.* Such a system possesses a natural classical limit state ω, which induces a classical limit system by the GNS construction. The criterion for quantum ergodicity is that the time average $\langle A \rangle$ of an observable $A \in \mathscr{A}$ equals the "space average" $\omega(A)I$ plus an error K which is negligible in the classical limit. We prove that ergodicity of ω is a sufficient condition for quantum ergodicity of $(\mathscr{A}, G, \alpha)$ if the classical limit system is abelian, give a conditional converse, and discuss a number of applications.

0. Introduction

The purpose of this paper is to generalize some basic notions and results on quantum ergodicity ([Sn, CV, Su, Z.1, Z.2]) to a wider class of C^* dynamical systems (A, G, α) which we call *quantized GNS systems* (Definition 1.1). The key feature of such a system is an invariant state ω which in a certain sense is the barycenter of the normal invariant states. By the Gelfand–Naimark–Segal construction, it induces a new system $(\mathscr{A}_\omega, G, \alpha_\omega)$, which will play the role of the classical limit. Our main abstract result (Theorem 1) shows that if $(\mathscr{A}, G, \alpha)$ is a quantized GNS system, if the classical limit is abelian (or if $(\mathscr{A}, \omega)$ is a "G-abelian" pair), and if ω is an ergodic state, then "almost all" the ergodic normal invariant states ρ_j of the system tend to ω as the "energy" $E(\rho_j) \to \infty$. This leads to an intrinsic notion of the quantum ergodicity of a quantized GNS system in terms of operator time and space averages (Definition 0.1), and to the result that a quantized GNS system is quantum ergodic if its classical limit is an ergodic abelian system (or if $(\mathscr{A}, \omega)$ is an ergodic G-abelian pair) (Theorem 2). Concrete applications include a simplified proof of quantum ergodicity of the wave group of a compact Riemannian manifold with ergodic geodesic flow, as well as extensions to manifolds with concave boundary and ergodic billiards, to quotient Hamiltonian systems on symplectic quotients and to ergodic Hamiltonian subsystems on sympletic subcones. More elaborate

★ Partially supported by NSF grant # DMS-9404637.