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Homoclinic Points in Symplectic and Volume-Preserving Diffeomorphisms

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Abstract: Let M^n be a compact n-dimensional manifold and ω be a symplectic or volume form on M^n. Let ϕ be a C^1 diffeomorphism on M^n that preserves ω and let p be a hyperbolic periodic point of ϕ. We show that generically p has a homoclinic point, and moreover, the homoclinic points of p is dense on both stable manifold and unstable manifold of p. Takens [11] obtained the same result for $n = 2$.

1. Introduction and Statement of Main Results

Let M^n be an n-dimensional compact manifold with a symplectic or volume structure. Recall that a volume structure is a non-degenerate differential n-form ω on M^n and a symplectic structure is a closed differential two-form ω such that $\omega \wedge \cdots \wedge \omega$ ($n/2$ times) is a non-degenerate volume form. A symplectic manifold is always even dimensional. We consider diffeomorphisms of M^n that preserve the differential form ω. A diffeomorphism that preserves symplectic (resp. volume) form ω is called a *symplectic* (resp. *volume-preserving*) diffeomorphism. Symplectic diffeomorphism arises naturally as a time-one map and the Poincaré map of Hamiltonian systems.

Let $\mathrm{Diff}^r_\omega(M^n)$ denote the set of C^r diffeomorphisms that preserve ω, with uniform C^r topology. Let $\phi \in \mathrm{Diff}^r_\omega(M^n)$, then $\phi^*(\omega) = \omega$. Let p be a point in M^n, p is said to be a *periodic point* of ϕ with period k if $\phi^k(p) = p$. Periodic points with period one are called *fixed points.* A periodic point p with period k is said to be *hyperbolic* if $d(\phi^k)|_{T_p(M^n)} : T_p(M^n) \to T_p(M^n)$ has no eigenvalue on the unit circle. For any hyperbolic periodic point p, there exist a *stable manifold,* denoted by $W^s_\phi(p)$ and an *unstable manifold,* denoted by $W^u_\phi(p)$. A *homoclinic point* of p with respect to ϕ is an intersection of $W^u_\phi(p)$ and $W^s_\phi(p)$, which differs from p, i.e., q is a homoclinic point of p if $q \in W^s_\phi(p) \cap W^s_\phi(p) \setminus \{p\}$. q is a *transversal* homoclinic point of p with respect to ϕ if $T_q(M) = T_q(W^s_\phi(p)) \oplus T_q(W^u_\phi(p))$.

We state our main results.

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