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Geometry of the Transport Equation in Multicomponent WKB Approximations

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Abstract: Although the WKB approximation for multicomponent systems has been intensively studied in the literature, its geometric and global aspects are much less well understood than in the scalar case. In this paper we give a completely geometric derivation of the transport equation, without using local sections and without assuming complete diagonalizability of the matrix valued principal symbol, or triviality of its eigenbundles. The term (unnamed in the previous literature) appearing in the transport equation in addition to the covariant derivative with respect to a natural projected connection is a tensor, independent of the choice of any sections. We give a geometric interpretation of this tensor, involving the contraction of the curvature of the eigenbundle and an analog of the second fundamental form with the Poisson tensor in phase space. In the non-degenerate case this term may be rewritten in an even simpler geometric form. Finally, we discuss obstructions to the existence of WKB states and give a geometric description of the quantization condition for WKB states for a non-degenerate eigenvalue-function.

1. Introduction

In its original analytic form, the so-called WKB method for obtaining asymptotic eigenfunctions for linear partial differential operators involves writing a trial approximate eigenfunction for an operator H in the form $\psi(x) = e^{iS(x)/\hbar}a(x)$. Expanding $H\psi - E\psi$ in powers of $\hbar$ leads first to a nonlinear first order partial differential equation (the eikonal, or Hamilton-Jacobi equation) for the phase function S and then to a linear homogeneous first order partial differential equation for the amplitude a (the transport equation).

A geometric version of the WKB method was developed by Maslov [16] and Hörmander [10], in which the phase function is represented by a lagrangian submanifold L in classical phase space, and the amplitude by a half-density α on L. This geometric approach makes it possible to extend the WKB method to cover in a natural

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