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Invariant Measures for the 2*D*-Defocusing Nonlinear Schrödinger Equation

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Abstract: Consider the $2D$ defocusing cubic NLS $iu_t + \Delta u - u|u|^2 = 0$ with Hamiltonian $\int(|\nabla\phi|^2 + \frac{1}{2}|\phi|^4)$. It is shown that the Gibbs measure constructed from the Wick ordered Hamiltonian, i.e. replacing $|\phi|^4$ by $:|\phi|^4:$, is an invariant measure for the appropriately modified equation $iu_t + \Delta u - [u|u|^2 - 2(\int |u|^2 dx)u] = 0$. There is a well defined flow on the support of the measure. In fact, it is shown that for almost all data ϕ the solution u, $u(0) = \phi$, satisfies $u(t) - e^{it\Delta}\phi \in C_{H^s}(\mathbb{R})$, for some $s > 0$. First a result local in time is established and next measure invariance considerations are used to extend the local result to a global one (cf. [B2]).

Introduction

Consider the Wick ordering $H_N = \int |\nabla u|^2 + \frac{1}{2}\int |u|^4 - 2a_N \int |u|^2 + a_N^2$ of the $2D$-Hamiltonian $\int |\nabla u|^2 + \frac{1}{2}\int |u|^4$ corresponding to the $2D$-defocusing cubic NLS.[1] It is shown that the solutions $u_N = u_N^\omega$ of the Cauchy problem

$$\begin{cases} (u_N)_t = i\frac{\partial H_N}{\partial \bar{u}} \equiv \Delta u_N - P_N(u_N|u_N|^2) + 2a_N u_N = 0 \\ u_N = P_N u_N\,, \quad u_N(0) = \sum_{|n| \leqq N} \frac{g_n(\omega)}{|n|} e^{i\langle x,n\rangle} \end{cases} \tag{i}$$

converge weakly for all time, for almost all ω.[2] Here $\{g_n(\omega) \mid n \in \mathbf{Z}\}$ are independent L^2-normalized complex Gaussians and P_N denotes the usual Dirichlet projection on the trigonometric system.

In fact, there is some $s > 0$, such that

$$u_N(t) - e^{2ic_N(\omega)t} \sum_{|n|<N} \frac{g_n(\omega)}{|n|} e^{i(\langle x,n\rangle + |n|^2 t)}\,, \tag{ii}$$

[1] u is a complex function.

[2] We ignore for notational simplicity the problem of the zero Fourier mode in (i). This problem may be avoided replacing $|n|$ by $(|n|^2 + \kappa)^{1/2}$, $\kappa > 0$ (redefining the Laplacian).